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# Hermite-hadamard type ınequalities for multiplicatively $s$-convex functions

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## Abstract

In this paper, some integral inequalities of Hermite-Hadamard type for multiplicatively $s$-convex functions are obtained. Also, some new inequalities involving multiplicative integrals are established for product and quotient of convex and multiplicatively $s$-convex functions.

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## 1. Introduction

Let $\mathfrak{I} \subset \mathbb{R}$ be an interval with $v_1, v_2 \in \mathfrak{I}$ and $v_1 < v_2$ and let $\psi: \mathfrak{I} \to \mathbb{R}$ be a convex function. The double inequality

$$\psi\left(\frac{v_1+v_2}{2}\right) \leq \frac{1}{v_2-v_1}\int_{v_1}^{v_2} \psi(x) \leq \frac{\psi(v_1)+\psi(v_2)}{2} \tag{1}$$

is known in the literature as Hermite-Hadamard integral inequality (see [1, 2]). In recent years, the inequalities (1) have received renewed attention and have grown into a significant tool for mathematical analysis, probability theory, optimization and other fields of mathematics. Also, by looking into diversity of settings, these inequalities are found to have a great number of uses.

The most celebrated inequality related to the integral mean of a convex function is the Hermite-Hadamard inequality. It has been studied extensively by a number of authors, since it was established by Hermite (1883) and Hadamard (1896), independently. For some extensions and generalizations of the Hermite-Hadamard inequality using novel and innovative methods, see [3-13] and the corresponding references cited therein.

### 1.1. Multiplicative calculus

Recall that the concept of multiplicative integral is denoted by $\int_{v_1}^{v_2}\left(\psi(x)\right)^{dx}$ while the ordinary integral is denoted by $\int_{v_1}^{v_2}\left(\psi(x)\right)dx$. This is because the sum of the terms of product is used in the definition of a classical Riemann integral of $\psi$ on $[v_1, v_2]$, the product of terms raised to certain powers is used in the definition of multiplicative integral of $\psi$ on $[v_1, v_2]$.

There is the following relation between Riemann integral and multiplicative integral [14].

**Proposition 1.1** If $\psi$ is positive and Riemann integrable on $[v_1, v_2]$, then $\psi$ is multiplicatively integrable on $[v_1, v_2]$ and

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$$\int_{v_1}^{v_2} \left(\psi(x)\right)^{dx} = e^{\int_{v_1}^{v_2} \ln(\psi(x))dx}.$$

In [14], Bashirov et al. show that multiplicative integral has the following results and notations:

**Proposition 1.2** If $\psi$ is positive and Riemann integrable on $[v_1, v_2]$, then $\psi$ is multiplicatively integrable on $[v_1, v_2]$ and

1. $\int_{v_1}^{v_2} \left(\left(\psi(x)\right)^p\right)^{dx} = \int_{v_1}^{v_2} \left(\left(\psi(x)\right)^{dx}\right)^p$,

2. $\int_{v_1}^{v_2} \left(\psi(x)\phi(x)\right)^{dx} = \int_{v_1}^{v_2} \left(\psi(x)\right)^{dx} . \int_{v_1}^{v_2} \left(\phi(x)\right)^{dx}$,

3. $\int_{v_1}^{v_2} \left(\frac{\psi(x)}{\phi(x)}\right)^{dx} = \frac{\int_{v_1}^{v_2} \left(\psi(x)\right)^{dx}}{\int_{v_1}^{v_2} \left(\phi(x)\right)^{dx}}$,

4. $\int_{v_1}^{v_2} \left(\psi(x)\right)^{dx} = \int_{v_1}^{\mu} \left(\psi(x)\right)^{dx} . \int_{\mu}^{v_2} \left(\psi(x)\right)^{dx}, \ v_1 \leq \mu \leq v_2$.

5. $\int_{v_1}^{v_1} \left(\psi(x)\right)^{dx} = 1$ and $\int_{v_1}^{v_2} \left(\psi(x)\right)^{dx} = \left(\int_{v_2}^{v_1} \left(\psi(x)\right)^{dx}\right)^{-1}$.

### 1.2. Preliminaries

Now, we will give some basic definitions and results.

**Definition 1.3 [2]** The function $\psi: [v_1, v_2] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality holds for all $x, y \in [v_1, v_2]$ and $\lambda \in [0,1]$:

$$\psi(\lambda x + (1-\lambda)y) \leq \lambda\psi(x) + (1-\lambda)\psi(y).$$

The function $\psi$ is said to be concave if $-\psi$ is convex.

**Definition 1.4 [2]** A function $\psi: \mathfrak{I} \to (0, \infty)$ is said to be multiplicatively or log convex, if

$$\psi((1-\lambda)x + \lambda y) \leq [\psi(x)]^{1-\lambda}[\psi(y)]^{\lambda}$$

for all $x, y \in \mathfrak{I}$ and $\lambda \in [0,1]$.

In [15], Ali et al. established Hermite-Hadamard inequality for multiplicatively convex functions as follows:

**Theorem 1.5** Let $\psi$ be a positive and multiplicatively convex function on interval $[v_1, v_2]$. Then

$$\psi\left(\frac{v_1+v_2}{2}\right) \leq \left(\int_{v_1}^{v_2} \left(\psi(x)\right)^{dx}\right)^{\frac{1}{v_2-v_1}} \leq G\big(\psi(v_1), \psi(v_2)\big),$$

where $G(.,.)$ is the geometric mean.

**Definition 1.6 [16]** A function $\psi: \mathfrak{I} \to (0, \infty)$ is said to be quasiconvex, if

$\psi((1-\lambda)x+\lambda y)\leq \max\{\psi(x),\psi(y)\}$

for all $x, y \in \mathfrak{I}$ and $\lambda \in [0,1]$.

From the above definitions we have

$$\psi\big((1-\lambda)x+\lambda y\big)\leq [\psi(x)]^{1-\lambda}[\psi(y)]^{\lambda}$$

$$\leq \psi(x)+\lambda[\psi(y)-\psi(x)]$$

$$\leq \max\{\psi(x),\psi(y)\}.$$

**Definition 1.7 [17]** A function $\psi:[0,\infty)\to\mathbb{R}$ is said to be $s$-convex in second sense if

$$\psi((1-\lambda)x+\lambda y)\leq (1-\lambda)^s\psi(x)+\lambda^s\psi(y)$$

holds for all $x, y \in [0,\infty)$, $\lambda \in [0,1]$ and $s \in (0,1]$.

**Remark 1.8** For $s=1$, Definition 1.7. reduces to Definition 1.3.

In [18], Dragomir and Fitzpatrick proved a variant of Hermite–Hadamard inequality for $s$-convex functions as follows:

**Theorem 1.9** Let $\psi:[0,\infty)\to(0,\infty)$ be an $s$-convex function in second sense, where $s\in(0,1]$. Let $v_1, v_2 \in (0,\infty)$ and $v_1<v_2$. If $\psi\in L[0,1]$, then the following inequality holds:

$$2^{s-1}\psi\left(\frac{v_1+v_2}{2}\right)\leq \frac{1}{v_2-v_1}\int_{v_1}^{v_2}\psi(x)dx\leq \frac{\psi(v_1)+\psi(v_2)}{s+1}. \tag{2}$$

**Remark 1.10** If we take $s=1$ in Theorem 1.9, then the inequality $(2)$ becomes to inequality $(1)$.

**Definition 1.11 [19]** A function $\psi:\mathfrak{I}\to(0,\infty)$ is said to be multiplicatively or log $s$-convex if

$$\psi((1-\lambda)x+\lambda y)\leq [\psi(x)]^{(1-\lambda)^s}[\psi(y)]^{\lambda^s}$$

holds for all $x, y \in \mathfrak{I}$, $\lambda\in[0,1]$ and $s\in(0,1)$.

**Remark 1.11** For $s=1$, Definition 1.11 reduces to Definition 1.4.

## 2. Main Results

In this section we obtain some Hermite-Hadamard type integral inequalities in the setting of multiplicative calculus for multiplicatively $s$-convex and convex positive functions.

**Theorem 2.1** Let $\psi$ be a positive and multiplicatively $s$-convex function on $[v_1, v_2]$. Then the following inequalities hold:

$$\left[\psi\left(\frac{v_1+v_2}{2}\right)\right]^{2^{s-1}} \leq \left(\int_{v_1}^{v_2} \left(\psi(x)\right)^{dx}\right)^{\frac{1}{v_2-v_1}} \leq [\psi(v_1)\psi(v_2)]^{1/(s+1)}. \tag{3}$$

(3) is called Hermite-Hadamard integral inequalities for multiplicatively $s$-convex functions.

**Proof.** If $\psi$ is a multiplicatively $s$-convex positive function, then we have

$$\ln\psi\left(\frac{v_1+v_2}{2}\right) = \ln\left(\psi\left(\frac{(1-\lambda)v_1+\lambda v_2+\lambda v_1+(1-\lambda)v_2}{2}\right)\right)$$

$$= \ln\left(\psi\left(\frac{(1-\lambda)v_1+\lambda v_2}{2}+\frac{\lambda v_1+(1-\lambda)v_2}{2}\right)\right)$$

$$\leq \ln\left(\left(\psi\big((1-\lambda)v_1+\lambda v_2\big)\right)^{\frac{1}{2^s}}.\left(\psi(\lambda v_1+(1-\lambda)v_2)\right)^{\frac{1}{2^s}}\right)$$

$$= \frac{1}{2^s}\left[\ln\left(\psi\big((1-\lambda)v_1+\lambda v_2\big)\right)+\ln\big(\psi(\lambda v_1+(1-\lambda)v_2)\big)\right].$$

Integrating the above inequality with respect to $\lambda$ on $[0,1]$, we get

$$\ln\psi\left(\frac{v_1+v_2}{2}\right) \leq \int_0^1 \frac{1}{2^s}\left[\ln\left(\psi\big((1-\lambda)v_1+\lambda v_2\big)\right)d\lambda+\ln\big(\psi(\lambda v_1+(1-\lambda)v_2)\big)\right]d\lambda$$

$$= \frac{1}{2^s}\left[\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\psi(x)\big)dx+\frac{1}{v_1-v_2}\int_{v_2}^{v_1}\ln\big(\psi(x)\big)dx\right]$$

$$= \frac{1}{2^s}\left[\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\psi(x)\big)dx+\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\psi(x)\big)dx\right]$$

$$= \frac{1}{2^{s-1}}.\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\psi(x)\big)dx,$$

which implies that

$$2^{s-1}\ln\psi\left(\frac{v_1+v_2}{2}\right) \leq \frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\psi(x)\big)dx.$$

Thus, we have

$$\left[\psi\left(\frac{v_1+v_2}{2}\right)\right]^{2^{s-1}} \leq e^{\left(\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln(\psi(x))dx\right)}$$

$$= \left(\int_{v_1}^{v_2} \left(\psi(x)\right)^{dx}\right)^{\frac{1}{v_2-v_1}}.$$

Hence, we obtain

$$\left[\psi\left(\frac{v_1+v_2}{2}\right)\right]^{2^{s-1}} \leq \left(\int_{v_1}^{v_2} \left(\psi(x)\right)^{dx}\right)^{\frac{1}{v_2-v_1}}, \tag{4}$$

which completes the proof of the left hand side of (3). Now consider the right hand side of (3).

$$\left(\int_{v_1}^{v_2} \left(\psi(x)\right)^{dx}\right)^{\frac{1}{v_2-v_1}} = \left(e^{\left(\int_{v_1}^{v_2} \ln(\psi(x))dx\right)}\right)^{\frac{1}{v_2-v_1}}$$

$$= e^{\frac{1}{v_2-v_1}\left(\int_{v_1}^{v_2} \ln(\psi(x))dx\right)}$$

$$= e^{\left(\int_0^1 \ln\left(\psi(v_1+\lambda(v_2-v_1))\right)d\lambda\right)}$$

$$\leq e^{\left(\int_0^1 \ln\left(\left(\psi(v_1)\right)^{(1-\lambda)^s}\left(\psi(v_2)\right)^{\lambda^s}\right)d\lambda\right)}$$

$$= e^{\left(\int_0^1 \left((1-\lambda)^s \ln\psi(v_1)+\lambda^s \ln\psi(v_2)\right)d\lambda\right)}$$

$$= e^{\left(\ln\left(\psi(v_1)\psi(v_2)\right)^{\int_0^1 \lambda^s d\lambda}\right)}$$

$$= [\psi(v_1)\psi(v_2)]^{1/(s+1)}$$

Hence, we get the inequality

$$\left(\int_{v_1}^{v_2} \left(\psi(x)\right)^{dx}\right)^{\frac{1}{v_2-v_1}} \leq [\psi(v_1)\psi(v_2)]^{1/(s+1)}. \tag{5}$$

Combining the inequalities (4) and (5), we have the inequality (3).

**Remark 2.2** If we choose $s = 1$, then Theorem 2.1 reduces to Theorem 1.5.

**Theorem 2.3** Let $\psi$ and $\phi$ be multiplicatively $s$-convex positive functions on $[v_1, v_2]$. Then the following inequalities hold:

$$\left[\psi\left(\frac{v_1+v_2}{2}\right)\phi\left(\frac{v_1+v_2}{2}\right)\right]^{2^{s-1}} \leq \left(\int_{v_1}^{v_2} \left(\psi(x)\right)^{dx} . \int_{v_1}^{v_2} \left(\phi(x)\right)^{dx}\right)^{\frac{1}{v_2-v_1}}$$

$$\leq \left[\left(\psi(v_1)\psi(v_2)\right).\left(\phi(v_1)\phi(v_2)\right)\right]^{1/(s+1)}. \tag{6}$$

**Proof.** Since $\psi$ and $\phi$ are multiplicatively $s$-convex positive functions, we have

$$\ln\left(\psi\left(\frac{v_1+v_2}{2}\right)\phi\left(\frac{v_1+v_2}{2}\right)\right) = \ln\left(\psi\left(\frac{v_1+v_2}{2}\right)\right) + \ln\left(\phi\left(\frac{v_1+v_2}{2}\right)\right)$$

$$= \ln\left(\psi\left(\frac{(1-\lambda)v_1+\lambda v_2+\lambda v_1+(1-\lambda)v_2}{2}\right)\right)$$

$$+\ln\left(\phi\left(\frac{(1-\lambda)v_1+\lambda v_2+\lambda v_1+(1-\lambda)v_2}{2}\right)\right)$$

$$= \ln\left(\psi\left(\frac{(1-\lambda)v_1+\lambda v_2}{2}+\frac{\lambda v_1+(1-\lambda)v_2}{2}\right)\right)$$

$$+\ln\left(\phi\left(\frac{(1-\lambda)v_1+\lambda v_2}{2}+\frac{\lambda v_1+(1-\lambda)v_2}{2}\right)\right)$$

$$\leq \ln\left(\left(\psi\left((1-\lambda)v_1+\lambda v_2\right)\right)^{\frac{1}{2^s}}.\left(\psi(\lambda v_1+(1-\lambda)v_2)\right)^{\frac{1}{2^s}}\right)$$

$$+\ln\left(\left(\phi\left((1-\lambda)v_1+\lambda v_2\right)\right)^{\frac{1}{2^s}}.\left(\phi(\lambda v_1+(1-\lambda)v_2)\right)^{\frac{1}{2^s}}\right)$$

$$= \frac{1}{2^s}\left[\ln\left(\psi\left((1-\lambda)v_1+\lambda v_2\right)\right)+\ln\left(\psi(\lambda v_1+(1-\lambda)v_2)\right)\right]$$

$$+\frac{1}{2^s}\left[\ln\left(\phi\left((1-\lambda)v_1+\lambda v_2\right)\right)+\ln\left(\phi(\lambda v_1+(1-\lambda)v_2)\right)\right].$$

Integrating the above inequality with respect to $\lambda$ on $[0,1]$, we have

$$\ln\left(\psi\left(\frac{v_1+v_2}{2}\right)\phi\left(\frac{v_1+v_2}{2}\right)\right) \leq \int_0^1 \frac{1}{2^s}\left[\ln\left(\psi\left((1-\lambda)v_1+\lambda v_2\right)\right)+\ln\left(\psi(\lambda v_1+(1-\lambda)v_2)\right)d\lambda\right]$$

$$+\int_0^1 \frac{1}{2^s}\left[\ln\left(\phi\left((1-\lambda)v_1+\lambda v_2\right)\right)+\ln\left(\phi(\lambda v_1+(1-\lambda)v_2)\right)d\lambda\right]$$

$$= \frac{1}{2^s}\left[\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\left(\psi(x)\right)dx+\frac{1}{v_1-v_2}\int_{v_2}^{v_1}\ln\left(\psi(x)\right)dx\right]$$

$$+\frac{1}{2^s}\left[\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\phi(x)\big)dx+\frac{1}{v_1-v_2}\int_{v_2}^{v_1}\ln\big(\phi(x)\big)dx\right]$$

$$=\frac{1}{2^{s-1}}\left[\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\psi(x)\big)dx+\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\phi(x)\big)dx\right],$$

which implies that

$$2^{s-1}\ln\left(\psi\left(\frac{v_1+v_2}{2}\right)\phi\left(\frac{v_1+v_2}{2}\right)\right)\leq\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\psi(x)\big)dx+\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\phi(x)\big)dx.$$

Thus, we have

$$\left[\psi\left(\frac{v_1+v_2}{2}\right)\phi\left(\frac{v_1+v_2}{2}\right)\right]^{2^{s-1}}\leq e^{\left(\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln(\psi(x))dx+\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln(\phi(x))dx\right)}$$

$$=\left(e^{\int_{v_1}^{v_2}\ln(\psi(x))dx+\int_{v_1}^{v_2}\ln(\phi(x))dx}\right)^{\frac{1}{v_2-v_1}}$$

$$=\left(e^{\int_{v_1}^{v_2}\ln(\psi(x))dx}.e^{\int_{v_1}^{v_2}\ln(\phi(x))dx}\right)^{\frac{1}{v_2-v_1}}$$

$$=\left(\int_{v_1}^{v_2}\big(\psi(x)\big)^{dx}.\int_{v_1}^{v_2}\big(\phi(x)\big)^{dx}\right)^{\frac{1}{v_2-v_1}}.$$

Hence, we attain

$$\left[\psi\left(\tfrac{v_1+v_2}{2}\right)\phi\left(\tfrac{v_1+v_2}{2}\right)\right]^{2^{s-1}}\leq\left(\int_{v_1}^{v_2}\big(\psi(x)\big)^{dx}.\int_{v_1}^{v_2}\big(\phi(x)\big)^{dx}\right)^{\frac{1}{v_2-v_1}}. \tag{7}$$

Consider the second inequality in (6):

$$\left(\int_{v_1}^{v_2}\big(\psi(x)\big)^{dx}.\int_{v_1}^{v_2}\big(\phi(x)\big)^{dx}\right)^{\frac{1}{v_2-v_1}}$$

$$=\left(e^{\int_{v_1}^{v_2}\ln(\psi(x))dx+\int_{v_1}^{v_2}\ln(\phi(x))dx}\right)^{\frac{1}{v_2-v_1}}$$

$$=\left(e^{(v_2-v_1)\left(\int_0^1\ln\left(\psi(v_1+\lambda(v_2-v_1))\right)d\lambda+\int_0^1\ln\left(\phi(v_1+\lambda(v_2-v_1))\right)d\lambda\right)}\right)^{\frac{1}{v_2-v_1}}$$

$$= e^{\int_0^1 \ln\left(\psi(v_1+\lambda(v_2-v_1))\right)d\lambda+\int_0^1 \ln\left(\phi(v_1+\lambda(v_2-v_1))\right)d\lambda}$$

$$\leq e^{\int_0^1 \ln\left((\psi(v_1))^{(1-\lambda)^s}(\psi(v_2))^{\lambda^s}\right)d\lambda+\int_0^1 \ln\left((\phi(v_1))^{(1-\lambda)^s}(\phi(v_2))^{\lambda^s}\right)d\lambda}$$

$$= e^{\int_0^1 \left((1-\lambda)^s\ln(\psi(v_1))+\lambda^s\ln(\psi(v_2))\right)d\lambda+\int_0^1 \left((1-\lambda)^s\ln(\phi(v_1))+\lambda^s\ln(\phi(v_2))\right)d\lambda}$$

$$= e^{\ln(\psi(v_1).\psi(v_2))^{\int_0^1 \lambda^s d\lambda}+\ln(\phi(v_1).\phi(v_2))^{\int_0^1 \lambda^s d\lambda}}$$

$$= \left[\left(\psi(v_1)\psi(v_2)\right).\left(\phi(v_1)\phi(v_2)\right)\right]^{1/(s+1)}.$$

Hence, we have

$$\left(\int_{v_1}^{v_2}(\psi(x))^{dx}.\int_{v_1}^{v_2}(\phi(x))^{dx}\right)^{\frac{1}{v_2-v_1}} \leq \left[\left(\psi(v_1)\psi(v_2)\right).\left(\phi(v_1)\phi(v_2)\right)\right]^{1/(s+1)}. \tag{8}$$

Combining the inequalities (7) and (8) completes the proof.

**Remark 2.4** If we choose $s = 1$, then Theorem 2.3 reduces to Theorem 7 in [15].

**Theorem 2.5** Let $\psi$ and $\phi$ be multiplicatively $s$-convex positive functions on $[v_1, v_2]$. Then the following inequalities hold:

$$\left[\frac{\psi\left(\frac{v_1+v_2}{2}\right)}{\phi\left(\frac{v_1+v_2}{2}\right)}\right]^{2^{s-1}} \leq \left(\frac{\int_{v_1}^{v_2}(\psi(x))^{dx}}{\int_{v_1}^{v_2}(\phi(x))^{dx}}\right)^{\frac{1}{v_2-v_1}} \leq \left[\frac{\psi(v_1)\psi(v_2)}{\phi(v_1)\phi(v_2)}\right]^{\frac{1}{s+1}}. \tag{9}$$

**Proof.** Since $\psi$ and $\phi$ are multiplicatively $s$-convex positive functions, we have

$$\ln\frac{\psi\left(\frac{v_1+v_2}{2}\right)}{\phi\left(\frac{v_1+v_2}{2}\right)} = \ln\left(\psi\left(\frac{v_1+v_2}{2}\right) - \phi\left(\frac{v_1+v_2}{2}\right)\right)$$

$$= \ln\left(\psi\left(\frac{v_1+\lambda(v_2-v_1)+v_2+\lambda(v_1-v_2)}{2}\right)\right)$$

$$-\ln\left(\phi\left(\frac{v_1+\lambda(v_2-v_1)+v_2+\lambda(v_1-v_2)}{2}\right)\right)$$

$$= \ln\left(\psi\left(\frac{v_1+\lambda(v_2-v_1)}{2}+\frac{v_2+\lambda(v_1-v_2)}{2}\right)\right)$$

$$-\ln\left(\phi\left(\frac{v_1+\lambda(v_2-v_1)}{2}+\frac{v_2+\lambda(v_1-v_2)}{2}\right)\right)$$

$$\leq \ln\left(\left(\psi\big(v_1+\lambda(v_2-v_1)\big)\right)^{\frac{1}{2^s}}.\left(\psi\big(v_2+\lambda(v_1-v_2)\big)\right)^{\frac{1}{2^s}}\right)$$

$$-\ln\left(\left(\phi\big(v_1+\lambda(v_2-v_1)\big)\right)^{\frac{1}{2^s}}.\left(\phi\big(v_2+\lambda(v_1-v_2)\big)\right)^{\frac{1}{2^s}}\right)$$

$$=\frac{1}{2^s}\left[\ln\left(\psi\big(v_1+\lambda(v_2-v_1)\big)\right)+\ln\left(\psi\big(v_2+\lambda(v_1-v_2)\big)\right)\right]$$

$$-\frac{1}{2^s}\left[\ln\left(\phi\big(v_1+\lambda(v_2-v_1)\big)\right)+\ln\left(\phi\big(v_2+\lambda(v_1-v_2)\big)\right)\right].$$

Integrating the above inequality with respect to $\lambda$ on $[0,1]$, we have

$$\ln\frac{\psi\left(\frac{v_1+v_2}{2}\right)}{\phi\left(\frac{v_1+v_2}{2}\right)} \leq \int_0^1 \frac{1}{2^s}\left[\ln\left(\psi\big(v_1+\lambda(v_2-v_1)\big)\right)+\ln\left(\psi\big(v_2+\lambda(v_1-v_2)\big)\right)d\lambda\right]$$

$$-\int_0^1 \frac{1}{2^s}\left[\ln\left(\phi\big(v_1+\lambda(v_2-v_1)\big)\right)+\ln\left(\phi\big(v_2+\lambda(v_1-v_2)\big)\right)d\lambda\right]$$

$$=\frac{1}{2^s}\left[\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\psi(x)\big)dx+\frac{1}{v_1-v_2}\int_{v_2}^{v_1}\ln\big(\psi(x)\big)dx\right]$$

$$-\frac{1}{2^s}\left[\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\phi(x)\big)dx+\frac{1}{v_1-v_2}\int_{v_2}^{v_1}\ln\big(\phi(x)\big)dx\right]$$

$$=\frac{1}{2^{s-1}}\left[\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\psi(x)\big)dx-\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\phi(x)\big)dx\right],$$

which is equivalent to

$$2^{s-1}\ln\frac{\psi\left(\frac{v_1+v_2}{2}\right)}{\phi\left(\frac{v_1+v_2}{2}\right)} \leq \frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\psi(x)\big)dx-\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln\big(\phi(x)\big)dx.$$

Thus, we have

$$\left[\frac{\psi\left(\frac{v_1+v_2}{2}\right)}{\phi\left(\frac{v_1+v_2}{2}\right)}\right]^{2^{s-1}} \leq e^{\left(\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln(\psi(x))dx-\frac{1}{v_2-v_1}\int_{v_1}^{v_2}\ln(\phi(x))dx\right)}$$

$$=\left(e^{\int_{v_1}^{v_2}\ln(\psi(x))dx-\int_{v_1}^{v_2}\ln(\phi(x))dx}\right)^{\frac{1}{v_2-v_1}}$$

$$=\left(\frac{e^{\int_{v_1}^{v_2}\ln(\psi(x))dx}}{e^{\int_{v_1}^{v_2}\ln(\phi(x))dx}}\right)^{\frac{1}{v_2-v_1}}$$

$$=\left(\frac{\int_{v_1}^{v_2}\left(\psi(x)\right)^{dx}}{\int_{v_1}^{v_2}\left(\phi(x)\right)^{dx}}\right)^{\frac{1}{v_2-v_1}}.$$

Hence,

$$\left[\frac{\psi\left(\frac{v_1+v_2}{2}\right)}{\phi\left(\frac{v_1+v_2}{2}\right)}\right]^{2^{s-1}} \leq \left(\frac{\int_{v_1}^{v_2}\left(\psi(x)\right)^{dx}}{\int_{v_1}^{v_2}\left(\phi(x)\right)^{dx}}\right)^{\frac{1}{v_2-v_1}}. \tag{10}$$

Now, consider the second inequality in (9).

$$\left(\frac{\int_{v_1}^{v_2}\left(\psi(x)\right)^{dx}}{\int_{v_1}^{v_2}\left(\phi(x)\right)^{dx}}\right)^{\frac{1}{v_2-v_1}} = \left(\frac{e^{\int_{v_1}^{v_2}\ln(\psi(x))dx}}{e^{\int_{v_1}^{v_2}\ln(\phi(x))dx}}\right)^{\frac{1}{v_2-v_1}}$$

$$=\left(e^{\int_{v_1}^{v_2}\ln(\psi(x))dx-\int_{v_1}^{v_2}\ln(\phi(x))dx}\right)^{\frac{1}{v_2-v_1}}$$

$$=\left(e^{\left(\int_0^1\ln\left(\psi\left(v_1+\lambda(v_2-v_1)\right)\right)d\lambda-\int_0^1\ln\left(\phi\left(v_1+\lambda(v_2-v_1)\right)\right)d\lambda\right)}\right)^{\frac{1}{v_2-v_1}}$$

$$=e^{\int_0^1\ln\left(\psi\left(v_1+\lambda(v_2-v_1)\right)\right)d\lambda-\int_0^1\ln\left(\phi\left(v_1+\lambda(v_2-v_1)\right)\right)d\lambda}$$

$$\leq e^{\int_0^1\ln\left(\left(\psi(v_1)\right)^{(1-\lambda)^s}\left(\psi(v_2)\right)^{\lambda^s}\right)d\lambda-\int_0^1\ln\left(\left(\phi(v_1)\right)^{(1-\lambda)^s}\left(\phi(v_2)\right)^{\lambda^s}\right)d\lambda}$$

$$=e^{\int_0^1\left((1-\lambda)^s\ln\psi(v_1)+\lambda^s\ln\psi(v_2)\right)d\lambda-\int_0^1\left((1-\lambda)^s\ln\phi(v_1)+\lambda^s\ln\phi(v_2)\right)d\lambda}$$

$$= e^{\ln(\psi(v_1).\psi(v_2))^{\int_0^1 \lambda^s d\lambda} - \ln(\phi(v_1).\phi(v_2))^{\int_0^1 \lambda^s d\lambda}}$$

$$= \left[\frac{\psi(v_1)\psi(v_2)}{\phi(v_1)\phi(v_2)}\right]^{\frac{1}{s+1}}.$$

Hence,

$$\left(\frac{\int_{v_1}^{v_2} (\psi(x))^{dx}}{\int_{v_1}^{v_2} (\phi(x))^{dx}}\right)^{\frac{1}{v_2-v_1}} \leq \left[\frac{\psi(v_1)\psi(v_2)}{\phi(v_1)\phi(v_2)}\right]^{\frac{1}{s+1}}. \tag{11}$$

Using the inequalities (10) and (11) gives the desired result.

**Remark 2.6** If we choose $s = 1$, then Theorem 2.5 reduces to Theorem 9 in [15].

**Theorem 2.7** Let $\psi$ and $\phi$ be convex and multiplicatively $s$-convex positive functions, respectively. Then, we have

$$\left(\frac{\int_{v_1}^{v_2} (\psi(x))^{dx}}{\int_{v_1}^{v_2} (\phi(x))^{dx}}\right)^{\frac{1}{v_2-v_1}} \leq \frac{\left(\frac{(\psi(v_2))^{\psi(v_2)}}{(\psi(v_1))^{\psi(v_1)}}\right)^{\frac{1}{\psi(v_2)-\psi(v_1)}}}{e.\left(\phi(v_1)\phi(v_2)\right)^{1/(s+1)}}.$$

**Proof.** Note that

$$\left(\frac{\int_{v_1}^{v_2} (\psi(x))^{dx}}{\int_{v_1}^{v_2} (\phi(x))^{dx}}\right)^{\frac{1}{v_2-v_1}} = \left(\frac{e^{\int_{v_1}^{v_2} \ln(\psi(x))dx}}{e^{\int_{v_1}^{v_2} \ln(\phi(x))dx}}\right)^{\frac{1}{v_2-v_1}}$$

$$= \left(e^{\int_{v_1}^{v_2} \ln(\psi(x))dx - \int_{v_1}^{v_2} \ln(\phi(x))dx}\right)^{\frac{1}{v_2-v_1}}$$

$$= e^{\int_0^1 \ln\left(\psi(v_1+\lambda(v_2-v_1))\right)d\lambda - \int_0^1 \ln\left(\phi(v_1+\lambda(v_2-v_1))\right)d\lambda}$$

$$\leq e^{\int_0^1 \ln\left(\psi(v_1)+\lambda(\psi(v_2)-\psi(v_1))\right)d\lambda - \int_0^1 \ln\left((\phi(v_1))^{(1-\lambda)^s}(\phi(v_2))^{\lambda^s}\right)d\lambda}$$

$$= e^{\ln\left(\left(\frac{(\psi(v_2))^{\psi(v_2)}}{(\psi(v_1))^{\psi(v_1)}}\right)^{\frac{1}{\psi(v_2)-\psi(v_1)}}\right) - 1 - \ln(\phi(v_1)\phi(v_2))^{\int_0^1 \lambda^s d\lambda}}$$

$$= \frac{\left(\frac{(\psi(v_2))^{\psi(v_2)}}{(\psi(v_1))^{\psi(v_1)}}\right)^{\frac{1}{\psi(v_2)-\psi(v_1)}}}{e.\left(\phi(v_1)\phi(v_2)\right)^{1/(s+1)}}.$$

Thus, we have

$$\left(\frac{\int_{v_1}^{v_2}(\psi(x))^{dx}}{\int_{v_1}^{v_2}(\phi(x))^{dx}}\right)^{\frac{1}{v_2-v_1}} \leq \frac{\left(\frac{(\psi(v_2))^{\psi(v_2)}}{(\psi(v_1))^{\psi(v_1)}}\right)^{\frac{1}{\psi(v_2)-\psi(v_1)}}}{e.\left(\phi(v_1)\phi(v_2)\right)^{1/(s+1)}},$$

which completes the proof.

**Remark 2.8** If we choose $s = 1$, then Theorem 2.7 reduces to Theorem 11 in [15].

**Theorem 2.9** Let $\psi$ and $\phi$ be multiplicatively $s$-convex and convex positive functions, respectively. Then, we have

$$\left(\frac{\int_{v_1}^{v_2}(\psi(x))^{dx}}{\int_{v_1}^{v_2}(\phi(x))^{dx}}\right)^{\frac{1}{v_2-v_1}} \leq \frac{e.\left(\psi(v_1)\psi(v_2)\right)^{1/(s+1)}}{\left(\frac{(\phi(v_2))^{\phi(v_2)}}{(\phi(v_1))^{\phi(v_1)}}\right)^{\frac{1}{\phi(v_2)-\phi(v_1)}}}.$$

**Proof.** Note that

$$\left(\frac{\int_{v_1}^{v_2}(\psi(x))^{dx}}{\int_{v_1}^{v_2}(\phi(x))^{dx}}\right)^{\frac{1}{v_2-v_1}} = \left(\frac{e^{\int_{v_1}^{v_2}\ln(\psi(x))dx}}{e^{\int_{v_1}^{v_2}\ln(\phi(x))dx}}\right)^{\frac{1}{v_2-v_1}}$$

$$= \left(e^{\int_{v_1}^{v_2}\ln(\psi(x))dx-\int_{v_1}^{v_2}\ln(\phi(x))dx}\right)^{\frac{1}{v_2-v_1}}$$

$$= e^{\int_0^1\ln\left(\psi(v_1+\lambda(v_2-v_1))\right)d\lambda-\int_0^1\ln\left(\phi(v_1+\lambda(v_2-v_1))\right)d\lambda}$$

$$\leq e^{\int_0^1\ln\left((\psi(v_1))^{(1-\lambda)^s}(\psi(v_2))^{\lambda^s}\right)d\lambda-\int_0^1\ln\left(\phi(v_1)+\lambda(\phi(v_2)-\phi(v_1))\right)d\lambda}$$

$$= e^{\ln(\psi(v_1)\psi(v_2))^{\int_0^1\lambda^s d\lambda}-\ln\left(\left(\frac{(\phi(v_2))^{\phi(v_2)}}{(\phi(v_1))^{\phi(v_1)}}\right)^{\frac{1}{\phi(v_2)-\phi(v_1)}}\right)+1}$$

$$= \frac{e.\left(\psi(v_1)\psi(v_2)\right)^{1/(s+1)}}{\left(\frac{(\phi(v_2))^{\phi(v_2)}}{(\phi(v_1))^{\phi(v_1)}}\right)^{\frac{1}{\phi(v_2)-\phi(v_1)}}}.$$

Hence,

$$\left(\frac{\int_{v_1}^{v_2}\left(\psi(x)\right)^{dx}}{\int_{v_1}^{v_2}\left(\phi(x)\right)^{dx}}\right)^{\frac{1}{v_2-v_1}} \le \frac{e.\left(\psi(v_1)\psi(v_2)\right)^{1/(s+1)}}{\left(\frac{(\phi(v_2))^{\phi(v_2)}}{(\phi(v_1))^{\phi(v_1)}}\right)^{\frac{1}{\phi(v_2)-\phi(v_1)}}},$$

which is the desired result.

**Remark 2.10** If we choose $s = 1$, then Theorem 2.9 reduces to Theorem 12 in [15].

**Theorem 2.11** Let $\psi$ and $\phi$ be convex and multiplicatively $s$-convex positive functions, respectively. Then, we have

$$\left(\int_{v_1}^{v_2}\left(\psi(x)\right)^{dx}.\int_{v_1}^{v_2}\left(\phi(x)\right)^{dx}\right)^{\frac{1}{v_2-v_1}} \le \frac{\left(\frac{(\psi(v_2))^{\psi(v_2)}}{(\psi(v_1))^{\psi(v_1)}}\right)^{\frac{1}{\psi(v_2)-\psi(v_1)}}.\left(\phi(v_1)\phi(v_2)\right)^{1/(s+1)}}{e}.$$

**Proof.** Note that

$$\left(\int_{v_1}^{v_2}\left(\psi(x)\right)^{dx}.\int_{v_1}^{v_2}\left(\phi(x)\right)^{dx}\right)^{\frac{1}{v_2-v_1}}$$

$$= \left(e^{\int_{v_1}^{v_2}\ln(\psi(x))dx+\int_{v_1}^{v_2}\ln(\phi(x))dx}\right)^{\frac{1}{v_2-v_1}}$$

$$= \left(e^{(v_2-v_1)\left(\int_0^1\ln\left(\psi(v_1+\lambda(v_2-v_1))\right)d\lambda+\int_0^1\ln\left(\phi(v_1+\lambda(v_2-v_1))\right)d\lambda\right)}\right)^{\frac{1}{v_2-v_1}}$$

$$= e^{\int_0^1\ln\left(\psi(v_1+\lambda(v_2-v_1))\right)d\lambda+\int_0^1\ln\left(\phi(v_1+\lambda(v_2-v_1))\right)d\lambda}$$

$$\le e^{\int_0^1\ln\left(\psi(v_1)+\lambda(\psi(v_2)-\psi(v_1))\right)d\lambda+\int_0^1\ln\left((\phi(v_1))^{(1-\lambda)^s}(\phi(v_2))^{\lambda^s}\right)d\lambda}$$

$$= e^{\ln\left(\left(\frac{(\psi(v_2))^{\psi(v_2)}}{(\psi(v_1))^{\psi(v_1)}}\right)^{\frac{1}{\psi(v_2)-\psi(v_1)}}\right)-1+\ln\left(\phi(v_1)\phi(v_2)\right)^{\int_0^1\lambda^s d\lambda}}$$

$$= \frac{\left(\frac{(\psi(v_2))^{\psi(v_2)}}{(\psi(v_1))^{\psi(v_1)}}\right)^{\frac{1}{\psi(v_2)-\psi(v_1)}} . \left(\phi(v_1)\phi(v_2)\right)^{1/(s+1)}}{e}.$$

Hence,

$$\left(\int_{v_1}^{v_2} \left(\psi(x)\right)^{dx} . \int_{v_1}^{v_2} \left(\phi(x)\right)^{dx}\right)^{\frac{1}{v_2-v_1}} \leq \frac{\left(\frac{(\psi(v_2))^{\psi(v_2)}}{(\psi(v_1))^{\psi(v_1)}}\right)^{\frac{1}{\psi(v_2)-\psi(v_1)}} . \left(\phi(v_1)\phi(v_2)\right)^{1/(s+1)}}{e}.$$

This completes the proof.

**Remark 2.12** If we choose $s = 1$, then Theorem 2.11 reduces to Theorem 13 in [15].

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