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Adomian decomposition method for solving nonlinear fractional sturm-liouville problem

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Abstract

In the present paper, the Adomian decomposition method is employed for solving nonlinear fractional Sturm-Liouville equation. The numerical results for the eigenfunctions and the eigenvalues are obtained. Also, the present results are demonstrated by the tables and the graphs for different values of considered problem.

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1. Introduction

The Adomian Decomposition Method (ADM) was presented by G. Adomian in [1]. Differential and integral equations of linear and nonlinear problems have been solved by this method [2-6]. By the ADM, the representation of the solution has been synchronized to the series. Each term of the series is calculated from a polynomial produced by a power series expansion of an analytic function. This method provides effective algorithms for the analytic approximate solution. In the last years, many researchers have studied the applications of ADM to solve various problems in [1, 7, 8, 19, 22]. This method can be employed for any kind of differential and integral equations, homogeneous or inhomogeneous linear or nonlinear, with constant coefficients or with variable coefficients. Also, the calculation of this method is easy and gives highly accurate numerical results.

The linear and nonlinear Sturm-Liouville (S-L) problems have of great importance. Theory and algorithms of these problems are presented in [9-18]. The fractional specific problems are studied in [20], [21].

The purpose of this paper is to understand the structure of eigenvalues and eigenfunctions for nonlinear S-L problem with fractional Riemann-Liouville derivative. To this end, we establish the values of $\lambda(\alpha)$, numerically for different order of α. We also analyzed the eigenfunctions corresponding to different eigenvalues and illustrate the results with figures and tables.

2. General Requirements

Definition 1. [10] Let $0<\alpha\le 1$. The right and left-sided Riemann-Liouville (RL) integrals of order α are introduced by

$$\left(I_{a,+}^{\alpha} f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1} f(s)\,ds,\quad x>a$$

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$$\left(I_{b,-}^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (s-x)^{\alpha-1}f(s)ds,\ \ x < b$$

where Γ denotes the gamma function.

Definition 2. [10] Let $0<\alpha\le 1$. The right and left-sided RL derivatives of order α are introduced by

$$\left(D_{a,+}^{\alpha}f\right)(x)=D\left(I_{a,+}^{1-\alpha}f\right)(x),\quad x>a$$

$$\left(D_{b,-}^{\alpha}f\right)(x)=-D\left(I_{b,-}^{1-\alpha}f\right)(x)\quad x<b.$$

Property 3. [10] If f(t) is continuous for $t\ge a$ then integration arbitrary of real order p and q RL integral has the following important property

$$I_{a,+}^{p}\left(I_{a,+}^{q}f\right)(x)=I_{a,+}^{p+q}f(x).$$

Lemma 4. [10] For $k\in\mathbb{N}, \alpha\in\mathbb{R}^{+}$ if $g\in L_1[a,b]$ and $k-1<\alpha<k$

$$D_{a,+}^{p}I_{a,+}^{q}g(x)=g(x),$$

$$I_{a,+}^{q}D_{a,+}^{p}g(x)=g(x)-\sum_{m=0}^{k-1}g^{(m)}(0+)\frac{(x-a)^m}{m!}$$

where $b>a\ge 0$ and $x\ge 0$.

3. Analysis Of The ADM

Consider the nonlinear fractional S-L problem

$$-D_{0+}^{\alpha}\left(\frac{d}{dx}y(x)\right)+e^{y(x)}=\lambda y(x),\quad x\in(0,1) \tag{1}$$

having following boundary conditions

$$y(0)=0\quad y(1)=0 \tag{2}$$

where $0\le\alpha\le 1$, $y(x)>0$ and $\lambda>0$ is a spectral parameter.

The purpose is the work is to investigate the eigenvalues and the eigenfunctions of $(1),(2)$ by using the ADM.

The equation (1) can be expressed as

$$Ly(x)=e^{y(x)}-\lambda y(x) \tag{3}$$

where $L=D_{0+}^{\alpha}\left(\frac{d}{dx}\right)$ is the differential operator. We can give the inverse of L by

$$L^{-1}[.]\ =\int_0^x I_{0+}^{\alpha}(.)dt.$$

Applying L^{-1} on the left side of equation (3) and using the initial condition $y(0)=0$, it can be obtained that

$$\begin{aligned}\left(L^{-1}L\right)\left(y(x)\right) &= \int_0^x I_{0+}^{\alpha}\left(D_{0+}^{\alpha}\frac{dy(t)}{dt}\right)dt \\ &= \int_0^x y'(t)-y'(0)\,dt \\ &= y(x)-y(0)-cx\end{aligned}$$

where $c=y'(0)\neq 0$.

Applying L^{-1} on the both side of equation (3), it gives that

$$y(x)=cx+L^{-1}\left(e^{y(x)}-\lambda y(x)\right) \tag{4}$$

By the ADM, $y(x)$ can be symbolized such that

$$y(x)=\sum_{n=0}^{\infty}y_n(x) \tag{5}$$

in here N is given by

$$e^{y(x)}=N\left(y(x,\lambda)\right)=\sum_{n=0}^{\infty}A_n \tag{6}$$

where A_n are the Adomian polynomials of $y_0, y_1, ..., y_n$ given by

$$A_n=\frac{1}{n!}\frac{d^n}{d\mu^n}\left[N\left(\sum_{i=0}^{\infty}\mu^i y_i(x)\right)\right]_{\mu=0},\ n=0,1,2,\ldots$$

Substituting (5), (6) into (4) gives

$$\sum_{n=0}^{\infty}y_n(x)=cx+L^{-1}\left(\sum_{n=0}^{\infty}A_n(x)\right)-L^{-1}\left(\lambda\sum_{n=0}^{\infty}y_n(x)\right). \tag{7}$$

The following recursive relations can be given from equation (7) :

$$\begin{aligned} y_0(x) &= cx \\ y_{k+1}(x) &= L^{-1}\left(A_k(x)-\lambda y_k\right),\ k\geq 0.\end{aligned} \tag{8}$$

$A_k(x)$ are the Adomian polynomials which are including the nonlinear term $N(y)=e^{y(x)}$ and following algorithm will be used for calculating Adomian polynomials

$$
\begin{aligned}
A_0(x) &= N(y_0) = e^{y_0}, \\
A_1(x) &= y_1 N'(y_0) = e^{y_0} y_1, \\
A_2(x) &= y_2 N'(y_0) + \frac{1}{2} y_1^2 N''(y_0) = e^{y_0} y_2 + \frac{1}{2} e^{y_0} y_1^2 \\
A_3(x) &= y_3 N'(y_0) + y_1 y_2 N''(y_0) + \frac{1}{3!} y_1^3 N'''(y_0) \\
&= e^{y_0} y_3 + e^{y_0} y_1 y_2 + \frac{e^{y_0}}{3!} y_1^3 \\
&. \\
&.
\end{aligned}
\tag{9}
$$

Combining (8) and (9) yields

$$y_0 = cx,$$

$$
\begin{aligned}
y_1 &= \int_0^x I_{0+}^{\alpha} \left(e^{y_0(t)} - \lambda y_0(t) \right) dt \\
&= \int_0^x I_{0+}^{\alpha} \left(e^{ct} - \lambda ct \right) dt = \frac{x^{1+\alpha} \left(2 + \alpha + c(x + \lambda x) \right)}{\Gamma(3+\alpha)}.
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= \int_0^x I_{0+}^{\alpha} \left(y_1(t) e^{ct} - \lambda y_1(t) \right) dt \\
&= \frac{x^{2+2\alpha} - 2(2+\alpha)(3+2\alpha)(\lambda-1)}{\Gamma(5+2\alpha)} +
\end{aligned}
$$

$$\frac{x^{2+2\alpha} \left(2(2+\alpha) c \left(3 + \alpha + (\lambda-2)\lambda \right) x - (3+\alpha) c^2 (\lambda-1) x^2 \right)}{\Gamma(5+2\alpha)}.$$

$$y_3 = \int_0^x I_{0+}^{\alpha} \left(y_2(t) e^{ct} - \lambda y_2(t) + \frac{1}{2} y_1^2(t) e^{ct} \right) dt.$$

Calculating more terms in the decomposition series is improved to convergence. According to this idea the solution $y(x)$ is approximately equal to

$$y(x) = y_0 + y_1 + y_2 + y_3$$

On the other hand, the boundary condition $y(1,\lambda) = 0$ gives nonlinear equation

$$F(\lambda, c) = 0 \tag{10}$$

From here we can obtain the branching diagram of the problem (1)-(2).
We know that $y(x)$ satisfies

$$y(z) = y(1-z),\ 0 \le z \le 1.$$

It's clear that

$$G(\lambda, c) = y'(0) - y'(1) = 0. \tag{11}$$

Considering the equations (10) and (11) together and solving this system numerically we obtain the values of λ and c for different values of α as follow

Table 1. Approximate eigenvalues under different orders of α

α	λ	c
0.5	0.99512	-837.2
	1	5.78291×10^{14}
	1.64427	-2.81885×10^{15}
	1.71517	94.0068
	10.6761	1.20394
0.6	0.997324	-1465.31
	1	-8.84498×10^{14}
	1.58721	2.17009×10^{15}
	1.64314	146.988
	11.6322	1.10646
0.7	0.998394	-2601.29
	1	5.79212×10^{15}
	1.53447	-2.74528×10^{17}
	1.57757	232.096
	12.707	0.965743
0.8	0.999052	-4679.72
	1	-1.3532×10^{15}
	1.48583	2.34141×10^{16}
	1.51854	369.062
	14.0689	0.794993
0.9	-120.561	-0.0440089
	0.998121	-2217.96
	1	-2.75938×10^{15}

Now we show the representations of the different solutions under different values by via Table 1. All data are obtained numerically corresponding to the first two terms of series expansion of exponential function.

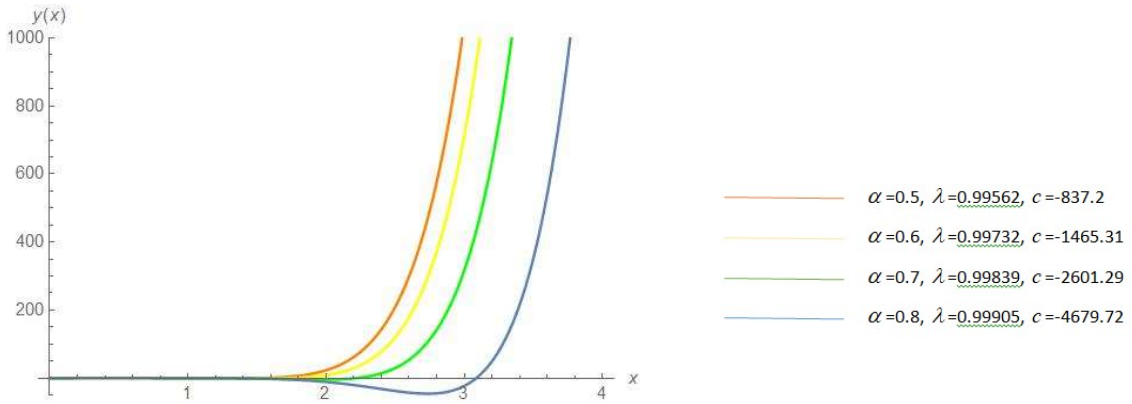

Figure 1. The representations of the solutions under different values

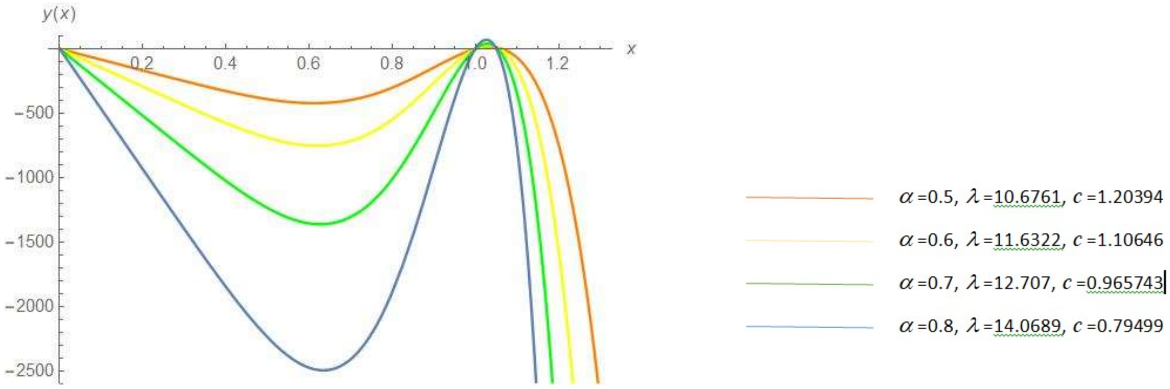

Figure 2. The representations of the solutions under different values

4. Conclusions

In this paper, nonlinear fractional S-L problem is considered. By employing the ADM, we analyze the eigenfunctions and eigenvalues. The results obtained in this paper demonstrate that ADM is a powerful method for finding the eigenvalues and eigenfunctions for nonlinear fractional S-L problem. This technique provides a convergent series solving the problem. These results new tools deal with the fractional differential and integral equations in mathematics, physics, biology etc.

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