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First order derivatives new h.hadamard type ınequalities for harmonically h convex functions

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Abstract

In this study, we derived a new integral identity for differentiable functions. However, we get new inequalities which is well known as Hermite-Hadamard (H-H) type by using the integral identity, which unifies the class of new and known harmonically convex functions. Moreover, in this study, the properties of first and second kind harmonically s-convex and harmonically s-Godunova-Levin functions are studied and some special cases are also dealt. Some important inferences are made at this study for supporting the results that obtained for classes of harmonically convex functions in previous studies.

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1. Introduction

Of late years, theory of convex functions has received special attention by many researchers on account of its importance in different fields of pure and applied sciences such as optimization and economics. Consequently the classical concepts of convex functions, see [1-4, 8, 9, 11, 13]. A significant generalization of convex functions was the introduction of h-convex functions by Varosanec [12], Noor [5] we introduced and investigate a new class of harmonically convex functions, which is called harmonically h-convex function and derived some new Hermite Hadamard like inequalities for harmonically h-convex functions.

In (Noor et al. 2015), the author gave definitions.

2. Materials and Methods

Definition 2.1 A function $k:\Omega\subset\mathbb{R}_+\to\mathbb{R}$ is said to be harmonically second kind of s-convex function, where $s\in(0,1]$, if

$$k\left(\frac{DE}{\nu D+(1-\nu)E}\right)\leq(1-\nu)^s k(D)+\nu^s k(E),\ \forall D,E\in I,\ \nu\in[0,1].$$

Definition 2.2 A function $k:\Omega\subset\mathbb{R}_+\to\mathbb{R}$ is said to be harmonically P-function, if

$$k\left(\frac{DE}{\nu D+(1-\nu)E}\right)\leq k(D)+k(E),\ \forall D,E\in I,\ \nu\in[0,1].$$

Definition 2.3 A function $k:\Omega\subset\mathbb{R}_+\to\mathbb{R}$ is said to be harmonically second kind of s-Godunova Levin function, if

$$k\left(\frac{DE}{\nu D+(1-\nu)E}\right)\leq\frac{1}{(1-\nu)^s}k(D)+\frac{1}{\nu^s}k(E)\ ,\ \forall D,E\in I,\ \nu\in(0,1),\ s\in[0,1].$$

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Definition 2.4 Let $h:[0,1]\subseteq J\to\mathbb{R}$ be a non-negative function. A function $k:\Omega\subset\mathbb{R}_+\to\mathbb{R}$ is said to be harmonically h-convex function, if

$$k\left(\frac{DE}{\nu D+(1-\nu)E}\right)\leq h(1-\nu)k(D)+h(\nu)k(E),\ \forall D,E\in I,\ \nu\in(0,1).$$

Definition 2.5 A function $k:\Omega\subset\mathbb{R}_+\to\mathbb{R}$ is said to be harmonically first kind of s-convex functions, where $s\in[0,1]$, if

$$k\left(\frac{DE}{\nu D+(1-\nu)E}\right)\leq(1-\nu^s)k(D)+\nu^s k(E),\ \forall D,E\in I,\ \nu\in[0,1].$$

Definition 2.6 A function $k:\Omega\subset\mathbb{R}_+\to\mathbb{R}$ is said to be harmonically first kind of s-Godunova Levin function, if

$$k\left(\frac{DE}{\nu D+(1-\nu)E}\right)\leq\frac{1}{(1-\nu)^s}k(D)+\frac{1}{\nu^s}k(E),\ \forall D,E\in I,\nu\in(0,1),s\in(0,1].$$

In [1], Iscan (2013), proved following Hermite Hadamard type inequality for harmonically convex functions.

Theorem 2.1 Let $k:\Omega\subset\mathbb{R}_+\to\mathbb{R}$ be harmonically convex function and $b,c\in\Omega$ with $b<c$. If $k\in L[b,c]$, then

$$k\left(\frac{2bc}{b+c}\right)\leq\frac{bc}{c-b}\int_b^c\frac{k(\mathrm{X})}{\mathrm{X}^2}d\mathrm{X}\leq\frac{k(b)+k(c)}{2}.$$

Lemma 2.1 Let $k:I\to\mathbb{R}$ be differentiable function on I^o (interior of I) and $b,c\in I$ with $b<c$. If $k'\in L[b,c]$, then

$$\frac{k(b)+k(c)}{2}-\frac{bc}{c-b}\int_b^c\frac{k(\mathrm{X})}{\mathrm{X}^2}d\mathrm{X}=\frac{bc(c-b)}{2}\int_0^1\frac{1-2\nu}{(\nu c+(1-\nu)b)^2}k'\left(\frac{bc}{\nu c+(1-\nu)b}\right)d\nu$$

Throughout this section, $h\left(\frac{1}{2}\right)\neq 0$, $I\subset\mathbb{R}_+$ be the interval and I^o be the interior of I, unless otherwise specified.

3. Main Results

Lemma 3.1 Let $k:I\to\mathbb{R}$ be second order differentiable function on I^o (interior of I) and $b,c\in I$ with $b<c$. If $k''\in L[b,c]$, then

$$k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]$$

$$+\frac{\lambda}{c\left(c-b\right)}k\left(c\right)-\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c\frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X}$$

$$=\int_0^{\frac{1}{2}}\frac{\nu\left(\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu+\int_{\frac{1}{2}}^1\frac{\left(1-\nu\right)\left(1-\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu$$

Proof. Let

$$I^*=\int_0^{\frac{1}{2}}\frac{\nu\left(\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu+\int_{\frac{1}{2}}^1\frac{\left(1-\nu\right)\left(1-\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu$$

By integrating by part, we have

$$I_1=\int_0^{\frac{1}{2}}\frac{\nu\left(\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu$$

$$I_2=\int_{\frac{1}{2}}^1\frac{\left(1-\nu\right)\left(1-\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu$$

With partial integration

$$I_1=k'\left(\frac{2bc}{b+c}\right)\left[\frac{1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]-k\left(\frac{2bc}{b+c}\right)\left[\frac{3b+c}{4b^2c^2\left(b-c\right)^2}+\frac{\lambda}{c\left(c-b\right)}\right]$$

$$+\frac{\lambda}{c\left(c-b\right)}k\left(c\right)+\frac{2}{\left(b-c\right)^3}\int_c^{\frac{2bc}{b+c}}\frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X}$$

Similarly, for

$$I_2=k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda-1}{2bc\left(b^2-c^2\right)}\right]+k\left(b\right)\left[\frac{-\lambda}{b^2c\left(b-c\right)^2}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{4c\lambda-b-3c}{4b^2c^2\left(b-c\right)^2}\right]+\frac{2}{\left(b-c\right)^3}\int_{\frac{2bc}{b+c}}^b\frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X}$$

When I_1 and I_2 are combined, the desired result is obtained. This completes the proof.

Now by Lemma 3.1, we prove our next results.

Theorem 3.1 Let $k:I\to\mathbb{R}$ be second order differentiable function on I^o where $b,c\in I$ with $b<c$ and $k''\in L\left[b,c\right]$. If $\left|k''\right|^q$, $q\geq 0$ is harmonically h-convex function, then, we have

$$\left| \begin{array}{l} k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) \\ -\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c\frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X} \end{array} \right|$$

$$\le \psi_1^{1-\frac{1}{q}}\left(\psi_2\left|k''\left(b\right)\right|^q+\psi_3\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}+\psi_4^{1-\frac{1}{q}}\left(\psi_5\left|k''\left(b\right)\right|^q+\psi_6\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}$$

Where

$$\psi_1=\int_0^{\frac{1}{2}}\left|\frac{\nu\left(\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}\right|d\nu \qquad \psi_2=\int_0^{\frac{1}{2}}\frac{\left|\nu\left(\nu-\lambda\right)\right|h\left(\nu\right)}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}d\nu$$

$$\psi_3=\int_0^{\frac{1}{2}}\frac{\left|\nu\left(\nu-\lambda\right)\right|h\left(1-\nu\right)}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}d\nu \qquad \psi_4=\int_{\frac{1}{2}}^{1}\left|\frac{\left(1-\nu\right)\left(1-\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}\right|d\nu \qquad \textbf{(3.1)}$$

$$\psi_5=\int_{\frac{1}{2}}^{1}\frac{\left|\left(1-\nu\right)\left(1-\nu-\lambda\right)\right|h\left(\nu\right)}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}d\nu \qquad \psi_6=\int_{\frac{1}{2}}^{1}\frac{\left|\left(1-\nu\right)\left(1-\nu-\lambda\right)\right|h\left(1-\nu\right)}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}d\nu$$

respectively.

Proof By Lemma 3.1, power mean inequality and the fact that $\left|k''\right|^q$ is harmonically h-convex function, we have

$$\left| \begin{array}{l} k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) \\ -\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c\frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X} \end{array} \right|$$

$$\le\left|\int_0^{\frac{1}{2}}\frac{\nu\left(\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu+\int_{\frac{1}{2}}^{1}\frac{\left(1-\nu\right)\left(1-\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu\right|$$

$$\le\left|\int_0^{\frac{1}{2}}\frac{\nu\left(\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu\right|+\left|\int_{\frac{1}{2}}^{1}\frac{\left(1-\nu\right)\left(1-\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu\right|$$

$$\leq \left(\int_0^{\frac{1}{2}} \left(\left| \frac{\nu(\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right|^{\frac{1}{p}} \right)^p d\nu \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left(\left| \frac{\nu(\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right|^{\frac{1}{q}} \left| k''\left(\frac{bc}{\nu c+(1-\nu)b} \right) \right| \right)^q d\nu \right)^{\frac{1}{q}}$$

$$+ \left(\int_{\frac{1}{2}}^{1} \left(\left| \frac{(1-\nu)(1-\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right|^{\frac{1}{p}} \right)^p d\nu \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \left(\left| \frac{(1-\nu)(1-\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right|^{\frac{1}{q}} \left| k''\left(\frac{bc}{\nu c+(1-\nu)b} \right) \right| \right)^q d\nu \right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^{\frac{1}{2}} \left| \frac{\nu(\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right| d\nu \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left| \frac{\nu(\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right| \left| k''\left(\frac{bc}{\nu c+(1-\nu)b} \right) \right|^q d\nu \right)^{\frac{1}{q}}$$

$$+ \left(\int_{\frac{1}{2}}^{1} \left| \frac{(1-\nu)(1-\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right| d\nu \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \left| \frac{(1-\nu)(1-\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right| \left| k''\left(\frac{bc}{\nu c+(1-\nu)b} \right) \right|^q d\nu \right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^{\frac{1}{2}} \left| \frac{\nu(\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right| d\nu \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \left| \frac{\nu(\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right| \left| k''\left(\frac{bc}{\nu c+(1-\nu)b} \right) \right|^q d\nu \right)^{\frac{1}{q}}$$

$$+ \left(\int_{\frac{1}{2}}^{1} \left| \frac{(1-\nu)(1-\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right| d\nu \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^{1} \left| \frac{(1-\nu)(1-\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right| \left| k''\left(\frac{bc}{\nu c+(1-\nu)b} \right) \right|^q d\nu \right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^{\frac{1}{2}} \left| \frac{\nu(\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right| d\nu \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \frac{|\nu(\nu-\lambda)|}{\left|(\nu c+(1-\nu)b)^3\right|} \begin{bmatrix} h(\nu)|k''(b)|^q \\ +h(1-\nu)|k''(c)|^q \end{bmatrix} d\nu \right)^{\frac{1}{q}}$$

$$+ \left(\int_{\frac{1}{2}}^{1} \left| \frac{(1-\nu)(1-\nu-\lambda)}{(\nu c+(1-\nu)b)^3} \right| d\nu \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^{1} \frac{|(1-\nu)(1-\nu-\lambda)|}{\left|(\nu c+(1-\nu)b)^3\right|} \begin{bmatrix} h(\nu)|k''(b)|^q \\ +h(1-\nu)|k''(c)|^q \end{bmatrix} d\nu \right)^{\frac{1}{q}}$$

$$\leq \psi_1^{1-\frac{1}{q}} \left(\psi_2 |k''(b)|^q + \psi_3 |k''(c)|^q \right)^{\frac{1}{q}} + \psi_4^{1-\frac{1}{q}} \left(\psi_5 |k''(b)|^q + \psi_6 |k''(c)|^q \right)^{\frac{1}{q}}$$

We get the result. This completes the proof.

Corollary 3.1 Under the conditions of Theorem 3.1, if $q=1$, then, we have

$$\left| k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) -\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c\frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X}\right|$$
$$\le\left(\psi_2\left|k''\left(b\right)\right|+\psi_3\left|k''\left(c\right)\right|\right)+\left(\psi_5\left|k''\left(b\right)\right|+\psi_6\left|k''\left(c\right)\right|\right)$$

Where $\psi_2,\psi_3,\psi_5,\psi_6$, (3.1) respectively.

If $h\left(\nu\right)=\nu^s$ in Theorem 3.1, we have the result for harmonically s-convex functions of second kind.

Corollary 3.2 Let $k:I\to\mathbb{R}$ be second order differentiable function on I^o where $b,c\in I$ with $b<c$ and $k''\in L\left[b,c\right]$. If $\left|k''\right|^q, q\ge 0$ is harmonically s-convex function of second kind, then

$$\left| k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) -\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c\frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X}\right|$$
$$\le\psi_1^{1-\frac{1}{q}}\left(\kappa_1\left|k''\left(b\right)\right|^q+\kappa_2\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}+\psi_4^{1-\frac{1}{q}}\left(\kappa_3\left|k''\left(b\right)\right|^q+\kappa_4\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}$$

Where ψ_1,ψ_4, (3.1) respectively and

$$\kappa_1=\int_0^{\frac{1}{2}}\frac{\left|\nu\left(\nu-\lambda\right)\right|\nu^s}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}d\nu \qquad \kappa_2=\int_0^{\frac{1}{2}}\frac{\left|\nu\left(\nu-\lambda\right)\right|\left(1-\nu\right)^s}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}d\nu$$
$$\kappa_3=\int_{\frac{1}{2}}^{1}\frac{\left|\left(1-\nu\right)\left(1-\nu-\lambda\right)\right|\nu^s}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}d\nu \qquad \kappa_4=\int_{\frac{1}{2}}^{1}\frac{\left|\left(1-\nu\right)\left(1-\nu-\lambda\right)\right|\left(1-\nu\right)^s}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}d\nu \tag{3.2}$$

Theorem 3.2 Let $k:I\to\mathbb{R}$ be second order differentiable function on I^o where $b,c\in I$ with $b<c$ and $k''\in L\left[b,c\right]$. If $\left|k''\right|^q$, $q\ge 0$ is harmonically s-convex function of first kind, then

$$\left| k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) -\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c\frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X}\right|$$
$$\le\psi_1^{1-\frac{1}{q}}\left(\kappa_1\left|k''\left(b\right)\right|^q+\left(\psi_1-\kappa_1\right)\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}+\psi_4^{1-\frac{1}{q}}\left(\kappa_3\left|k''\left(b\right)\right|^q+\left(\psi_4-\kappa_3\right)\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}$$

Where ψ_1, ψ_4, (3.1) and κ_1,κ_4, (3.2) respectively.

Proof. By Lemma 3.1, power mean inequality and the fact that $|k''|^q$ is harmonically s-convex function of first kind, we have

$$\left| k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) -\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c\frac{k\left(X\right)}{X^3}dX \right|$$

$$\leq\left|\int_0^{\frac{1}{2}}\frac{\nu\left(\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu+\int_{\frac{1}{2}}^1\frac{\left(1-\nu\right)\left(1-\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu\right|$$

$$\leq\left|\int_0^{\frac{1}{2}}\frac{\nu\left(\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu\right|+\left|\int_{\frac{1}{2}}^1\frac{\left(1-\nu\right)\left(1-\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu\right|$$

$$\leq\left(\int_0^{\frac{1}{2}}\frac{\left|\nu\left(\nu-\lambda\right)\right|}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}d\nu\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\frac{\left|\nu\left(\nu-\lambda\right)\right|}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}\left|k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)\right|^q d\nu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1\frac{\left|\left(1-\nu\right)\left(1-\nu-\lambda\right)\right|}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}d\nu\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\frac{\left|\left(1-\nu\right)\left(1-\nu-\lambda\right)\right|}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}\left|k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)\right|^q d\nu\right)^{\frac{1}{q}}$$

$$\leq\left(\int_0^{\frac{1}{2}}\frac{\left|\nu\left(\nu-\lambda\right)\right|}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}d\nu\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\frac{\left|\nu\left(\nu-\lambda\right)\right|}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}\left[\begin{array}{l}\nu^s\left|k''\left(b\right)\right|^q\\+\left(1-\nu^s\right)\left|k''\left(c\right)\right|^q\end{array}\right]d\nu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1\frac{\left|\left(1-\nu\right)\left(1-\nu-\lambda\right)\right|}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}d\nu\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\frac{\left|\left(1-\nu\right)\left(1-\nu-\lambda\right)\right|}{\left|\left(\nu c+\left(1-\nu\right)b\right)^3\right|}\left[\begin{array}{l}\nu^s\left|k''\left(b\right)\right|^q\\+\left(1-\nu^s\right)\left|k''\left(c\right)\right|^q\end{array}\right]d\nu\right)^{\frac{1}{q}}$$

$$\leq\psi_1^{1-\frac{1}{q}}\left(\kappa_1\left|k''\left(b\right)\right|^q+\left(\psi_1-\kappa_1\right)\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}+\psi_4^{1-\frac{1}{q}}\left(\kappa_3\left|k''\left(b\right)\right|^q+\left(\psi_4-\kappa_3\right)\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}$$

This completes the proof.

If $h(\nu)=1$ in Theorem 3.1, we have the result for harmonically, P-functions.

Corollary 3.3 Let $k:I\rightarrow\mathbb{R}$ be second order differentiable function on I^o where $b,c\in I$ with $b<c$ and $k''\in L[b,c]$. If $|k''|^q, q\geq 0$ is harmonically P-function, then

$$\left| \begin{array}{l} k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) \\ -\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c \frac{k\left(X\right)}{X^3}dX \end{array} \right|$$

$$\leq \psi_1^{1-\frac{1}{q}}\left(\left|k''(b)\right|^q+\left|k''(c)\right|^q\right)^{\frac{1}{q}}+\psi_4^{1-\frac{1}{q}}\left(\left|k''(b)\right|^q+\left|k''(c)\right|^q\right)^{\frac{1}{q}}$$

Where ψ_1, ψ_4, (3.1) respectively.

If $h(v)=v^{-s}$ in Theorem 3.1, we have the result for harmonically s -Godunova Levin functions of second kind

Corollary 3.4 Let $k: I \to \mathbb{R}$ be second order differentiable function on I^o where $b,c \in I$ with $b<c$ and $k'' \in L[b,c]$. If $|k''|^q, q \geq 0$ is harmonically s -Godunova Levin function of second kind, then

$$\left| \begin{array}{l} k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) \\ -\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c \frac{k\left(X\right)}{X^3}dX \end{array} \right|$$

$$\leq \psi_1^{1-\frac{1}{q}}\left(\lambda_1\left|k''(b)\right|^q+\lambda_2\left|k''(c)\right|^q\right)^{\frac{1}{q}}+\psi_4^{1-\frac{1}{q}}\left(\lambda_3\left|k''(b)\right|^q+\lambda_4\left|k''(c)\right|^q\right)^{\frac{1}{q}}$$

Where ψ_1, ψ_4, (3.1) respectively and

$$\begin{array}{ll} \lambda_1=\int_0^{\frac{1}{2}} \frac{\left|v(v-\lambda)\right|v^{-s}}{\left|\left(vc+(1-v)b\right)^3\right|}dv & \lambda_2=\int_0^{\frac{1}{2}} \frac{\left|v(v-\lambda)\right|(1-v)^{-s}}{\left|\left(vc+(1-v)b\right)^3\right|}dv \\ \lambda_3=\int_{\frac{1}{2}}^1 \frac{\left|(1-v)(1-v-\lambda)\right|v^{-s}}{\left|\left(vc+(1-v)b\right)^3\right|}dv & \lambda_4=\int_{\frac{1}{2}}^1 \frac{\left|(1-v)(1-v-\lambda)\right|(1-v)^{-s}}{\left|\left(vc+(1-v)b\right)^3\right|}dv \end{array} \qquad \textbf{(3.3)}$$

Theorem 3.3 Let $k: I \to \mathbb{R}$ be second order differentiable function on I^o where $b,c \in I$ with $b<c$ and $k'' \in L[b,c]$. If $|k''|^q, q \geq 0$ is harmonically s -Godunova Levin function of first kind, then

$$\left| \begin{array}{l} k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) \\ -\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c \frac{k\left(X\right)}{X^3}dX \end{array} \right|$$

$$\leq \psi_1^{1-\frac{1}{q}}\left(\lambda_1\left|k''(b)\right|^q+\lambda_2^*\left|k''(c)\right|^q\right)^{\frac{1}{q}}+\psi_4^{1-\frac{1}{q}}\left(\lambda_3\left|k''(b)\right|^q+\lambda_4^*\left|k''(c)\right|^q\right)^{\frac{1}{q}}$$

Where ψ_1, ψ_4, (3.1) and λ_1, λ_4, (3.3) respectively and

$$\lambda_2^* = \int_0^{\frac{1}{2}} \frac{|\nu(\nu-\lambda)|}{\left|(\nu c+(1-\nu)b)^3\right|(1-\nu^s)} d\nu$$

$$\lambda_4^* = \int_{\frac{1}{2}}^{1} \frac{|(1-\nu)(1-\nu-\lambda)|}{\left|(\nu c+(1-\nu)b)^3\right|(1-\nu^s)} d\nu$$

Proof. By Lemma 3.1, power mean inequality and the fact that $|k''|^q$ is harmonically s-Godunova Levin function, we have

$$\left| k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc(b^2-c^2)} - \frac{\lambda}{b+c}\right] + k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2(b-c)^2}\right] + \frac{\lambda}{c(c-b)}k(c) - \frac{\lambda}{b^2c(b-c)^2}k(b) + \frac{2}{(c-b)^3}\int_b^c \frac{k(X)}{X^3}dX \right|$$

$$\leq \left| \int_0^{\frac{1}{2}} \frac{\nu(\nu-\lambda)}{(\nu c+(1-\nu)b)^3} k''\left(\frac{bc}{\nu c+(1-\nu)b}\right) d\nu + \int_{\frac{1}{2}}^{1} \frac{(1-\nu)(1-\nu-\lambda)}{(\nu c+(1-\nu)b)^3} k''\left(\frac{bc}{\nu c+(1-\nu)b}\right) d\nu \right|$$

$$\leq \left| \int_0^{\frac{1}{2}} \frac{\nu(\nu-\lambda)}{(\nu c+(1-\nu)b)^3} k''\left(\frac{bc}{\nu c+(1-\nu)b}\right) d\nu \right| + \left| \int_{\frac{1}{2}}^{1} \frac{(1-\nu)(1-\nu-\lambda)}{(\nu c+(1-\nu)b)^3} k''\left(\frac{bc}{\nu c+(1-\nu)b}\right) d\nu \right|$$

$$\leq \left(\int_0^{\frac{1}{2}} \frac{|\nu(\nu-\lambda)|}{\left|(\nu c+(1-\nu)b)^3\right|} d\nu \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \frac{|\nu(\nu-\lambda)|}{\left|(\nu c+(1-\nu)b)^3\right|} \left|k''\left(\frac{bc}{\nu c+(1-\nu)b}\right)\right|^q d\nu \right)^{\frac{1}{q}}$$

$$+ \left(\int_{\frac{1}{2}}^{1} \frac{|(1-\nu)(1-\nu-\lambda)|}{\left|(\nu c+(1-\nu)b)^3\right|} d\nu \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^{1} \frac{|(1-\nu)(1-\nu-\lambda)|}{\left|(\nu c+(1-\nu)b)^3\right|} \left|k''\left(\frac{bc}{\nu c+(1-\nu)b}\right)\right|^q d\nu \right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^{\frac{1}{2}} \frac{|\nu(\nu-\lambda)|}{\left|(\nu c+(1-\nu)b)^3\right|} d\nu \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \frac{|\nu(\nu-\lambda)|}{\left|(\nu c+(1-\nu)b)^3\right|} \left[\frac{1}{\nu^s}|k''(b)|^q + \frac{1}{(1-\nu^s)}|k''(c)|^q \right] d\nu \right)^{\frac{1}{q}}$$

$$+ \left(\int_{\frac{1}{2}}^{1} \frac{|(1-\nu)(1-\nu-\lambda)|}{\left|(\nu c+(1-\nu)b)^3\right|} d\nu \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^{1} \frac{|(1-\nu)(1-\nu-\lambda)|}{\left|(\nu c+(1-\nu)b)^3\right|} \left[\frac{1}{\nu^s}|k''(b)|^q + \frac{1}{(1-\nu^s)}|k''(c)|^q \right] d\nu \right)^{\frac{1}{q}}$$

$$\leq \psi_1^{1-\frac{1}{q}} \left(\lambda_1 |k''(b)|^q + \lambda_2^* |k''(c)|^q \right)^{\frac{1}{q}} + \psi_4^{1-\frac{1}{q}} \left(\lambda_3 |k''(b)|^q + \lambda_4^* |k''(c)|^q \right)^{\frac{1}{q}}$$

This completes the proof.

Theorem 3.4 Let $k: I \to \mathbb{R}$ be second order differentiable function on I^o where $b,c \in I$ with $b<c$ and $k'' \in L[b,c]$. If $|k''|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$, is harmonically h-convex. Then we have

$$\left| k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) -\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c\frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X}\right|$$

$$\leq \psi_7^{\frac{1}{p}}\left(\psi_8\left|k''\left(b\right)\right|^q+\psi_9\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}+\psi_{10}^{\frac{1}{p}}\left(\psi_{11}\left|k''\left(b\right)\right|^q+\psi_{12}\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}$$

Where

$$\psi_7=\int_0^{\frac{1}{2}}\left|\nu\left(\nu-\lambda\right)\right|^p d\nu \qquad \psi_8=\int_0^{\frac{1}{2}}\frac{h\left(\nu\right)}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|}d\nu$$

$$\psi_9=\int_0^{\frac{1}{2}}\frac{h\left(1-\nu\right)}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|}d\nu \qquad \psi_{10}=\int_{\frac{1}{2}}^1\left|\left(1-\nu\right)\left(1-\nu-\lambda\right)\right|^p d\nu \qquad \mathbf{(3.4)}$$

$$\psi_{11}=\int_{\frac{1}{2}}^1\frac{h\left(\nu\right)}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|}d\nu \qquad \psi_{12}=\int_{\frac{1}{2}}^1\frac{h\left(1-\nu\right)}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|}d\nu$$

respectively.

Proof. By Lemma 3.1, Holder's inequality and the fact that $|k''|^q$ is harmonically h-convex function, we have

$$\left| k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) -\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c\frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X}\right|$$

$$\leq\left|\int_0^{\frac{1}{2}}\frac{\nu\left(\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu+\int_{\frac{1}{2}}^1\frac{\left(1-\nu\right)\left(1-\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu\right|$$

$$\leq\left|\int_0^{\frac{1}{2}}\frac{\nu\left(\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu\right|+\left|\int_{\frac{1}{2}}^1\frac{\left(1-\nu\right)\left(1-\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu\right|$$

$$\leq\left(\int_0^{\frac{1}{2}}\left|\nu(\nu-\lambda)\right|^p d\nu\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\frac{1}{\left|(\nu c+(1-\nu)b)^{3q}\right|}\left|k''\left(\frac{bc}{\nu c+(1-\nu)b}\right)\right|^q d\nu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|(1-\nu)(1-\nu-\lambda)\right|^p d\nu\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\frac{1}{\left|(\nu c+(1-\nu)b)^{3q}\right|}\left|k''\left(\frac{bc}{\nu c+(1-\nu)b}\right)\right|^q d\nu\right)^{\frac{1}{q}}$$

$$\leq\left(\int_0^{\frac{1}{2}}\left|\nu(\nu-\lambda)\right|^p d\nu\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\frac{1}{\left|(\nu c+(1-\nu)b)^{3q}\right|}\left\{h(\nu)\left|k''(b)\right|^q+h(1-\nu)\left|k''(c)\right|^q\right\}d\nu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|(1-\nu)(1-\nu-\lambda)\right|^p d\nu\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\frac{1}{\left|(\nu c+(1-\nu)b)^{3q}\right|}\left\{h(\nu)\left|k''(b)\right|^q+h(1-\nu)\left|k''(c)\right|^q\right\}d\nu\right)^{\frac{1}{q}}$$

$$\leq\psi_7^{\frac{1}{p}}\left(\psi_8\left|k''(b)\right|^q+\psi_9\left|k''(c)\right|^q\right)^{\frac{1}{q}}+\psi_{10}^{\frac{1}{p}}\left(\psi_{11}\left|k''(b)\right|^q+\psi_{12}\left|k''(c)\right|^q\right)^{\frac{1}{q}}$$

This completes the proof.

If $h(\nu)=\nu^s$ in Theorem 3.4, we have results for harmonically s-convex functions of second kind.

Corollary 3.5 Let $k:I\to\mathbb{R}$ be second order differentiable function on I^o where $b,c\in I$ with $b<c$ and $k''\in L[b,c]$. If $|k''|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$ is harmonically s-convex function of second kind. Then we have

$$\left|k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc(b^2-c^2)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2(b-c)^2}\right]+\frac{\lambda}{c(c-b)}k(c)-\frac{\lambda}{b^2c(b-c)^2}k(b)+\frac{2}{(c-b)^3}\int_b^c\frac{k(X)}{X^3}dX\right|$$

$$\leq\psi_7^{\frac{1}{p}}\left(\vartheta_1\left|k''(b)\right|^q+\vartheta_2\left|k''(c)\right|^q\right)^{\frac{1}{q}}+\psi_{10}^{\frac{1}{p}}\left(\vartheta_3\left|k''(b)\right|^q+\vartheta_4\left|k''(c)\right|^q\right)^{\frac{1}{q}}$$

Where ψ_7,ψ_{10}, (3.4) respectively and

$$\vartheta_1=\int_0^{\frac{1}{2}}\frac{\nu^s}{\left|(\nu c+(1-\nu)b)^{3q}\right|}d\nu\qquad\vartheta_2=\int_0^{\frac{1}{2}}\frac{(1-\nu)^s}{\left|(\nu c+(1-\nu)b)^{3q}\right|}d\nu$$

$$\vartheta_3=\int_{\frac{1}{2}}^{1}\frac{\nu^s}{\left|(\nu c+(1-\nu)b)^{3q}\right|}d\nu\qquad\vartheta_4=\int_{\frac{1}{2}}^{1}\frac{(1-\nu)^s}{\left|(\nu c+(1-\nu)b)^{3q}\right|}d\nu$$

Theorem 3.5 Let $k: I \to \mathbb{R}$ be second order differentiable function on I^o where $b, c \in I$ with $b < c$ and $k'' \in L[b,c]$. If $|k''|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$ is harmonically s-convex function of first kind. Then we have

$$\left| k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c(c-b)}k(c) \right.$$
$$\left. -\frac{\lambda}{b^2c(b-c)^2}k(b)+\frac{2}{(c-b)^3}\int_b^c \frac{k(X)}{X^3}dX \right|$$
$$\leq \psi_7^{\frac{1}{p}}\left(\vartheta_1^*\left|k''(b)\right|^q+\vartheta_2^*\left|k''(c)\right|^q\right)^{\frac{1}{q}}+\psi_{10}^{\frac{1}{p}}\left(\vartheta_3^*\left|k''(b)\right|^q+\vartheta_4^*\left|k''(c)\right|^q\right)^{\frac{1}{q}}$$

Where ψ_7, ψ_{10}, (3.4) respectively and

$$\vartheta_1^*=\int_0^{\frac{1}{2}}\frac{\nu^s}{\left|\left(\nu c+(1-\nu)b\right)^{3q}\right|}d\nu \qquad \vartheta_2^*=\int_0^{\frac{1}{2}}\frac{\left(1-\nu^s\right)}{\left|\left(\nu c+(1-\nu)b\right)^{3q}\right|}d\nu$$
$$\vartheta_3^*=\int_{\frac{1}{2}}^{1}\frac{\nu^s}{\left|\left(\nu c+(1-\nu)b\right)^{3q}\right|}d\nu \qquad \vartheta_4^*=\int_{\frac{1}{2}}^{1}\frac{\left(1-\nu^s\right)}{\left|\left(\nu c+(1-\nu)b\right)^{3q}\right|}d\nu$$

Proof. By Lemma 3.1, Holder's inequality and the fact that $|k''|^q$ is harmonically s-convex function of first kind, we have

$$\left| k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c(c-b)}k(c) \right.$$
$$\left. -\frac{\lambda}{b^2c(b-c)^2}k(b)+\frac{2}{(c-b)^3}\int_b^c \frac{k(X)}{X^3}dX \right|$$
$$\leq \left|\int_0^{\frac{1}{2}}\frac{\nu(\nu-\lambda)}{\left(\nu c+(1-\nu)b\right)^3}k''\left(\frac{bc}{\nu c+(1-\nu)b}\right)d\nu+\int_{\frac{1}{2}}^{1}\frac{(1-\nu)(1-\nu-\lambda)}{\left(\nu c+(1-\nu)b\right)^3}k''\left(\frac{bc}{\nu c+(1-\nu)b}\right)d\nu\right|$$
$$\leq \left|\int_0^{\frac{1}{2}}\frac{\nu(\nu-\lambda)}{\left(\nu c+(1-\nu)b\right)^3}k''\left(\frac{bc}{\nu c+(1-\nu)b}\right)d\nu\right|+\left|\int_{\frac{1}{2}}^{1}\frac{(1-\nu)(1-\nu-\lambda)}{\left(\nu c+(1-\nu)b\right)^3}k''\left(\frac{bc}{\nu c+(1-\nu)b}\right)d\nu\right|$$

$$\leq \left(\int_0^{\frac{1}{2}} \left|\nu(\nu-\lambda)\right|^p d\nu\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \frac{1}{\left|(\nu c+(1-\nu)b)^{3q}\right|} \left|k''\left(\frac{bc}{\nu c+(1-\nu)b}\right)\right|^q d\nu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1} \left|(1-\nu)(1-\nu-\lambda)\right|^p d\nu\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \frac{1}{\left|(\nu c+(1-\nu)b)^{3q}\right|} \left|k''\left(\frac{bc}{\nu c+(1-\nu)b}\right)\right|^q d\nu\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^{\frac{1}{2}} \left|\nu(\nu-\lambda)\right|^p d\nu\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \frac{1}{\left|(\nu c+(1-\nu)b)^{3q}\right|} \left\{\begin{array}{l} \nu^s \left|k''(b)\right|^q \\ +(1-\nu^s)\left|k''(c)\right|^q \end{array}\right\} d\nu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1} \left|(1-\nu)(1-\nu-\lambda)\right|^p d\nu\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \frac{1}{\left|(\nu c+(1-\nu)b)^{3q}\right|} \left\{\begin{array}{l} \nu^s \left|k''(b)\right|^q \\ +(1-\nu^s)\left|k''(c)\right|^q \end{array}\right\} d\nu\right)^{\frac{1}{q}}$$

$$\leq \psi_7^{\frac{1}{p}} \left(\vartheta_1^* \left|k''(b)\right|^q + \vartheta_2^* \left|k''(c)\right|^q\right)^{\frac{1}{q}} + \psi_{10}^{\frac{1}{p}} \left(\vartheta_3^* \left|k''(b)\right|^q + \vartheta_4^* \left|k''(c)\right|^q\right)^{\frac{1}{q}}$$

This completes the proof.

If $h(\nu)=1$ in Theorem 3.4, we have the result for harmonically P-functions.

Corollary 3.6 Let $k: I \to \mathbb{R}$ be second order differentiable function on I^o where $b,c \in I$ with $b<c$ and $k'' \in L[b,c]$. If $|k''|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$ is harmonically P-function, then we have

$$\left| \begin{array}{l} k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc(b^2-c^2)} - \frac{\lambda}{b+c}\right] + k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2(b-c)^2}\right] + \frac{\lambda}{c(c-b)}k(c) \\ -\frac{\lambda}{b^2c(b-c)^2}k(b) + \frac{2}{(c-b)^3}\int_b^c \frac{k(\mathrm{X})}{\mathrm{X}^3} d\mathrm{X} \end{array} \right|$$

$$\leq \psi_7^{\frac{1}{p}} \psi_1^* \left(\left|k''(b)\right|^q + \left|k''(c)\right|^q\right)^{\frac{1}{q}} + \psi_{10}^{\frac{1}{p}} \psi_2^* \left(\left|k''(b)\right|^q + \left|k''(c)\right|^q\right)^{\frac{1}{q}}$$

Where ψ_7, ψ_{10}, (3.4) respectively and

$$\alpha_1^* = \int_0^{\frac{1}{2}} \frac{1}{\left|(\nu c+(1-\nu)b)^{3q}\right|} d\nu$$

$$\alpha_2^* = \int_{\frac{1}{2}}^{1} \frac{1}{\left|(\nu c+(1-\nu)b)^{3q}\right|} d\nu$$

If $h(\nu)=\nu^{-s}$ in Theorem 3.4, we have the result for harmonically s-Godunova Levin functions of second kind.

Corollary 3.7 Let $k: I \rightarrow \mathbb{R}$ be second order differentiable function on I^o where $b,c \in I$ with $b<c$ and $k'' \in L[b,c]$. If $|k''|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$ is harmonically s-Godunova Levin function of second kind. Then, we have

$$\left| k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) - \frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c \frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X} \right|$$
$$\leq \psi_7^{\frac{1}{p}}\left(\varphi_1\left|k''\left(b\right)\right|^q+\varphi_2\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}+\psi_{10}^{\frac{1}{p}}\left(\varphi_3\left|k''\left(b\right)\right|^q+\varphi_4\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}$$

Where ψ_7, ψ_{10}, (3.4) respectively and

$$\varphi_1=\int_0^{\frac{1}{2}}\frac{\nu^{-s}}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|}d\nu \qquad \varphi_2=\int_0^{\frac{1}{2}}\frac{\left(1-\nu\right)^{-s}}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|}d\nu$$
$$\varphi_3=\int_{\frac{1}{2}}^{1}\frac{\nu^{-s}}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|}d\nu \qquad \varphi_4=\int_{\frac{1}{2}}^{1}\frac{\left(1-\nu\right)^{-s}}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|}d\nu \qquad \textbf{(3.5)}$$

Theorem 3.6 Let $k: I \rightarrow \mathbb{R}$ be second order differentiable function on I^o where $b,c \in I$ with $b<c$ and $k'' \in L[b,c]$. If $|k''|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$ harmonically s-Godunova Levin function of second kind. Then, we have

$$\left| k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) - \frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c \frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X} \right|$$
$$\leq \psi_7^{\frac{1}{p}}\left(\varphi_1\left|k''\left(b\right)\right|^q+\varphi_1^*\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}+\psi_{10}^{\frac{1}{p}}\left(\varphi_3\left|k''\left(b\right)\right|^q+\varphi_3^*\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}$$

Where ψ_7, ψ_{10} (3.4) and φ_1, φ_3, (3.5) respectively and

$$\varphi_1^*=\int_0^{\frac{1}{2}}\frac{1}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|\left(1-\nu^s\right)}d\nu$$

$$\varphi_2^*=\int_{\frac{1}{2}}^{1}\frac{1}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|\left(1-\nu^s\right)}d\nu$$

Proof. By Lemma 3.1, Holder's inequality and the fact that $|k''|^q$ is harmonically s-Godunova Levin function of second kind, we have

$$\left| k'\left(\frac{2bc}{b+c}\right)\left[\frac{2\lambda+1}{2bc\left(b^2-c^2\right)}-\frac{\lambda}{b+c}\right]+k\left(\frac{2bc}{b+c}\right)\left[\frac{c\lambda-c-b}{b^2c^2\left(b-c\right)^2}\right]+\frac{\lambda}{c\left(c-b\right)}k\left(c\right) -\frac{\lambda}{b^2c\left(b-c\right)^2}k\left(b\right)+\frac{2}{\left(c-b\right)^3}\int_b^c\frac{k\left(\mathrm{X}\right)}{\mathrm{X}^3}d\mathrm{X}\right|$$

$$\leq\left|\int_0^{\frac{1}{2}}\frac{\nu\left(\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu+\int_{\frac{1}{2}}^1\frac{\left(1-\nu\right)\left(1-\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu\right|$$

$$\leq\left|\int_0^{\frac{1}{2}}\frac{\nu\left(\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu\right|+\left|\int_{\frac{1}{2}}^1\frac{\left(1-\nu\right)\left(1-\nu-\lambda\right)}{\left(\nu c+\left(1-\nu\right)b\right)^3}k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)d\nu\right|$$

$$\leq\left(\int_0^{\frac{1}{2}}\left|\nu\left(\nu-\lambda\right)\right|^p d\nu\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\frac{1}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|}\left|k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)\right|^q d\nu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1\left|\left(1-\nu\right)\left(1-\nu-\lambda\right)\right|^p d\nu\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\frac{1}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|}\left|k''\left(\frac{bc}{\nu c+\left(1-\nu\right)b}\right)\right|^q d\nu\right)^{\frac{1}{q}}$$

$$\leq\left(\int_0^{\frac{1}{2}}\left|\nu\left(\nu-\lambda\right)\right|^p d\nu\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\frac{1}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|}\left\{\frac{1}{\nu^s}\left|k''\left(b\right)\right|^q+\frac{1}{\left(1-\nu^s\right)}\left|k''\left(c\right)\right|^q\right\}d\nu\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1\left|\left(1-\nu\right)\left(1-\nu-\lambda\right)\right|^p d\nu\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\frac{1}{\left|\left(\nu c+\left(1-\nu\right)b\right)^{3q}\right|}\left\{\frac{1}{\nu^s}\left|k''\left(b\right)\right|^q+\frac{1}{\left(1-\nu^s\right)}\left|k''\left(c\right)\right|^q\right\}d\nu\right)^{\frac{1}{q}}$$

$$\leq\psi_7^{\frac{1}{p}}\left(\varphi_1\left|k''\left(b\right)\right|^q+\varphi_1^*\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}+\psi_{10}^{\frac{1}{p}}\left(\varphi_3\left|k''\left(b\right)\right|^q+\varphi_3^*\left|k''\left(c\right)\right|^q\right)^{\frac{1}{q}}$$

This completes the proof.

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