Cumhuriyet Science Journal
e-ISSN: 2587-246X
ISSN: 2587-2680
Cumhuriyet Sci. J., 41(1) (2020) 1-10
http://dx.doi.org/10.17776/csj.613058

On complex gaussian jacobsthal and jacobsthal-lucas quaternions

Hasan ARSLAN

Erciyes University, Faculty of Science, Department of Mathematics, 38039, Kayseri / TURKEY

Abstract

The main aim of this work is to introduce the complex Gaussian Jacobsthal and Jacobsthal-Lucas quaternions and investigate their structures. We obtain the recurrence relations, Binet formulas and generating functions for these quaternions. We also give their Cassini identities by using Binet formulas. Furthermore, we prove some results for these quaternions such as summation formulas.

2010 *Mathematics Subject Classification:* 11R52; 11B39.

Article info

History:
Received:29.08.2019
Accepted:09.03.2020
Keywords:
Recurrence relation, Jacobsthal and Jacobsthal-Lucas numbers, Gaussian numbers, Quaternions.

1. Introduction

Real quaternions firstly observed by Sir William Rowan Hamilton in 1843, see [1]. The set of all real quaternions **H** is defined in the following way:

$$\mathbf{H}=\{q=q_0+q_1e_1+q_2e_2+q_3e_3 : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where $e_1^2=e_2^2=e_3^2=e_1e_2e_3=-1$. The conjugate of a quaternion is defined as $q^*=q_0-q_1e_1-q_2e_2-q_3e_3$. Quaternions have many applications in various research areas such as computer sciences, differential geometry, quantum physics, kinematic and analysis. Further information about quaternions see [2-8].

Horadam [9] introduced the Jacobsthal and Jacobsthal-Lucas numbers J_n and j_n which is given by the following recurrence relations

$$J_n=J_{n-1}+2J_{n-2},\ n\geq 2,\ J_0=0 \text{ and } J_1=1 \quad (1)$$

and

$$j_n=j_{n-1}+2j_{n-2},\ n\geq 2,\ j_0=2 \text{ and } j_1=1, \quad (2)$$

respectively. Recurrences (1) and (2) lead to the characteristic equation $t^2-t-2=0$ with roots $\alpha=2$ and $\beta=-1$ so that $\alpha+\beta=1$, $\alpha-\beta=3$ and $\alpha\beta=-2$. It is well-known from [9] that Binet formulas for the Jacobsthal and Jacobsthal-Lucas numbers are

$$J_n=\frac{2^n-(-1)^n}{3} \quad (3)$$

and

$$j_n=2^n+(-1)^n \quad (4)$$

respectively. Using the equations (3) and (4), we can deduce the extension of Jacobsthal and Jacobsthal-Lucas numbers to negative values of n such that $J_{-n}=\frac{(-1)^{n+1}}{2^n}J_n$ and $j_{-n}=\frac{(-1)^n}{2^n}j_n$.

In [10], Horadam gave many identities for Jacobsthal and Jacobsthal-Lucas numbers as follows:

$$2J_{n+1}=J_n+j_n \quad (5)$$

$$j_n=J_{n+1}+2J_{n-1} \quad (6)$$

*Corresponding author. *Email address: hasanarslan@erciyes.edu.tr*

$3J_n + j_n = 2^{n+1}$ $j_n J_n = J_{2n}$
$j_{m+n} = j_n J_{m+1} + 2 j_{n-1} J_m$
$2J_{m+n} = J_m j_n + j_m J_n$
$j_{n+1} j_{n-1} - j_n^2 = 9(-1)^{n-1} 2^{n-1} = -9(J_{n+1} J_{n-1} - J_n^2)$ (Simson formulas)
$\lim_{n\to\infty} \frac{J_{n+1}}{J_n} = \lim_{n\to\infty} \frac{j_{n+1}}{j_n} = 2$
$\lim_{n\to\infty} \frac{j_n}{J_n} = 3$
$\sum_{i=2}^{n} J_i = \frac{J_{n+2} - 3}{2}$
$\sum_{i=1}^{n} j_i = \frac{j_{n+2} - 5}{2}$.

There are several researches on quaternions whose coefficients consist of Fibonacci-like numbers, see [11-19].

As seen in literature, many authors contributed the theories of the Jacobsthal numbers, polynomials and Jacobsthal quaternions, see [9-12], [19], [20-25].
In [25], some identities for Jacobsthal numbers were given as follows:

$$J_n J_{n+1} + 2J_{n-1} J_n = J_{2n} = J_n j_n \tag{7}$$
$$J_n J_{m+1} + 2J_{n-1} J_m = J_{m+n}$$
$$J_{2n+1} = J_{n+1}^2 + 2J_n^2 \tag{8}$$
$$J_n J_{m-1} - J_{n-1} J_m = (-1)^m 2^{m-1} J_{n-m} .$$

In [19], Szynal-Liana and Wloch introduced Jacobsthal quaternion QJ_n and Jacobsthal-Lucas quaternion Qj_n in the following way:

$$QJ_n = J_n + J_{n+1} e_1 + J_{n+2} e_2 + J_{n+3} e_3 \tag{9}$$

and

$$Qj_n = j_n + j_{n+1} e_1 + j_{n+2} e_2 + j_{n+3} e_3 \tag{10}$$

respectively. They showed that these quaternions satisfy a second-order linear recurrence relation and obtained their norms and some relations between them.
In [11], Aydın Torunbalcı and Yüce explicitly determined Binet formulas of QJ_n and Qj_n as follows:

$$QJ_n = \frac{1}{\alpha - \beta}\left(A\alpha^n - B\beta^n\right) \text{ and } Qj_n = 3A\alpha^n + 3B\beta^n \tag{11}$$

respectively, where $\alpha - \beta = 3$, $A = 1 + 2e_1 + 4e_2 + 8e_3$, $B = 1 - e_1 + e_2 - e_3$.
They obtained the Cassini identities for QJ_n and Qj_n by using Binet formulas given in the equation (11).

In [26], Jordan defined the Gaussian Fibonacci and Gaussian Lucas numbers. Pethe and Horadam [27] defined Generalized Gaussian Fibonacci Numbers. In [20], Aşçı and Gürel introduced the Gaussian Jacobsthal and Gaussian Jacobsthal-Lucas sequences with numbers which satisfy the recurrence relations

$$GJ_n = GJ_{n-1} + 2GJ_{n-2},\ n \ge 2,\ GJ_0 = \frac{i}{2} \text{ and } GJ_1 = 1 \tag{12}$$

and

$$Gj_n = Gj_{n-1} + 2Gj_{n-2},\ n \ge 2,\ Gj_0 = 2 - \frac{i}{2} \text{ and } Gj_1 = 1 + 2i, \tag{13}$$

respectively. Here it can be easily seen that the relations $GJ_n = J_n + iJ_{n-1},\ n \ge 1$ and $Gj_n = j_n + ij_{n-1},\ n \ge 1$ exist. Also they gave some results relating Gaussian Jacobsthal and Jacobsthal-Lucas numbers as follows:

$$GJ_n = \frac{Gj_{n+1} + 2Gj_{n-1}}{9}$$
$$GJ_{n-1} GJ_{n+11} - GJ_n^2 = (3 - i)(-1)^n 2^{n-2}$$

$$Gj_{n-1}Gj_{n+11} - Gj_n^2 = 9(3-i)(-1)^{n-1}2^{n-2}$$

$$\sum_{k=0}^{n} GJ_n = \frac{1}{2}[GJ_{n+2} - 1]$$

$$\sum_{k=0}^{n} Gj_n = \frac{1}{2}[Gj_{n+2} - (1+2i)]$$

From (3) and (4), one can find Binet Formulas of GJ_n and Gj_n as $GJ_n = \frac{2^{n-1}(2+i)-(-1)^{n-1}(i-1)}{3}$ and $Gj_n = 2^{n-1}(2+i)+(-1)^{n-1}(i-1)$.

In this paper, we firstly introduce the quaternions whose coefficients Gaussian Jacobsthal and Gaussian Jacobsthal-Lucas numbers and then we give Binet formulas, Cassini identities, generating functions and some summation formulas for them.

2. Complex Gaussian Jacobsthal And Jacobsthal-LucasQuaternions

Any complex quaternion λ is defined in the following form

$\lambda = \lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$

where each coefficient λ_i is a complex number and e_1, e_2, e_3 are quaternionic units.

The set of all complex quaternions is denoted by H_C. The complex quaternion λ can be written as

$\lambda = q + iq'$

where q and q' are real quaternions. For further information see [28]. Complex quaternions, that is biquaternions are used in various research area such as the theory of special relativity, quantum mechanics, electromagnetism and particle physics, see [2-8]. Halıcı introduced the complex Fibonacci quaternions and examined their structures in [16] and Aydin Torunbalci introduced the generalized complex Jacobsthal sequence in [12].

Now, we define complex Gaussian Jacobsthal quaternion QGJ_n as follows:

$QGJ_n = GJ_n + GJ_{n+1}e_1 + GJ_{n+2}e_2 + GJ_{n+3}e_3 = QJ_n + iQJ_{n-1}$

where

$e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1,\ e_1e_2 = -e_2e_1 = e_3,\ e_2e_3 = -e_3e_2 = e_1,\ e_3e_1 = -e_1e_3 = e_2$.

Similarly, we define complex Gaussian Jacobsthal-Lucas quaternion QGj_n as

$QGj_n = Gj_n + Gj_{n+1}e_1 + Gj_{n+2}e_2 + Gj_{n+3}e_3 = Qj_n + iQj_{n-1}$.

Taking into account the equations $GJ_n = J_n + iJ_{n-1},\ n \geq 1$ and $Gj_n = j_n + ij_{n-1},\ n \geq 1$, the quaternions QGJ_n and QGj_n are expressible as $QGJ_n = QJ_n + iQJ_{n-1}$ and $QGj_n = Qj_n + iQj_{n-1}$, where QJ_n and Qj_n are n^{th} real Jacobsthal and Jacobsthal-Lucas quaternions as in (9) and (10), respectively. The addition, substraction and multiplication of the Gaussian Jacobsthal quaternions are defined as in real quaternions.

We state the quaternion conjugate of QGJ_n as

$QGJ_n^* = GJ_n - GJ_{n+1}e_1 - GJ_{n+2}e_2 - GJ_{n+3}e_3$,

and the complex conjugate of QGJ_n as

$\overline{QGJ_n} = \overline{GJ_n} + \overline{GJ_{n+1}}e_1 + \overline{GJ_{n+2}}e_2 + \overline{GJ_{n+3}}e_3$,

where $\overline{GJ_{i+r}}, r = 0,1,2,3$ stands for complex conjugate of the Gaussian Jacobsthal number $\overline{GJ_{i+r}}$. Thus, we conclude that $\overline{QGJ_n} = QJ_n - iQJ_{n-1}$.

Proposition 2.1. Let GJ_n be the n^{th} Gaussian Jacobsthal number. Then we have

$GJ_n^2 + 2GJ_{n-1}^2 = GJ_{2n-1} + iGJ_{2n-2}$.

Proof. Since $GJ_n = J_n + iJ_{n-1}$ and the equations (7) and (8), we get

$$\begin{aligned} GJ_n^{\,2}+2GJ_{n-1}^{\,2} &= \left(J_n+iJ_{n-1}\right)^2+2\left(J_{n-1}+iJ_{n-2}\right)^2 \\ &= \left(J_n^{\,2}+2J_{n-1}^{\,2}\right)-\left(J_{n-1}^{\,2}+2J_{n-2}^{\,2}\right)+2i\left(J_nJ_{n-1}+2J_{n-1}J_{n-2}\right) \\ &= J_{2n-1}-J_{2n-3}+2iJ_{2n-2} \\ &= J_{2n-1}+iJ_{2n-2}+iJ_{2n-2}-J_{2n-3} \\ &= GJ_{2n-1}+iGJ_{2n-2}. \end{aligned}$$

In the following proposition, we establish a few basic properties of the QGJ_n, some of which we will use to investigate the structure of QGJ_n in the rest of this section.

Proposition 2.2. Let n be a positive integer. Then we have

1. $QGJ_{n+1}=QGJ_n+2QGJ_{n-1}$ (14)
2. $QGj_{n+1}=QGj_n+2QGj_{n-1}$ (15)
3. $QGJ_n+QGJ_n^{\,*}=2GJ_n$ (16)
4. $QGJ_n+\overline{QGJ_n}=2QJ_n$ (17)
5. $QGJ_n^{\,2}+QGJ_n\left(QGJ_n^{\,*}\right)=2\left(QGJ_n\right)GJ_n$ (18)
6. $QGJ_n-QGJ_{n+1}e_1-QGJ_{n+2}e_2-QGJ_{n+3}e_3=\dfrac{2^{n-1}\cdot 85\left(2+i\right)-4\left(-1\right)^{n-1}\left(i-1\right)}{3}$.

Proof.

1. Using the equation (12) and the definition of Gaussian Jacobsthal quaternion, we get
$$\begin{aligned} QGJ_n+2QGJ_{n-1} &= \left(GJ_n+GJ_{n+1}e_1+GJ_{n+2}e_2+GJ_{n+3}e_3\right)+2\left(GJ_{n-1}+GJ_ne_1+GJ_{n+1}e_2+GJ_{n+2}e_3\right) \\ &= \left(GJ_n+2GJ_{n-1}\right)+\left(GJ_{n+1}+2GJ_n\right)e_1+\left(GJ_{n+2}+2GJ_{n+1}\right)e_2+\left(GJ_{n+3}+2GJ_{n+2}\right)e_3 \\ &= GJ_{n+1}+GJ_{n+2}e_1+GJ_{n+3}e_2+GJ_{n+4}e_3 \\ &= QGJ_{n+1} \end{aligned}$$
2. When used the equation (13) and the definition of QGj_n, the proof immediately follows.
3. From the definition of Gaussian Jacobsthal quaternion and its quaternion conjugate, we obtain
$QGJ_n+QGJ_n^{\,*}=\left(GJ_n+GJ_{n+1}e_1+GJ_{n+2}e_2+GJ_{n+3}e_3\right)+\left(GJ_n-GJ_{n+1}e_1-GJ_{n+2}e_2-GJ_{n+3}e_3\right)=2GJ_n$.
4. If we use the definition of Gaussian Jacobsthal quaternion and its complex conjugate, then we get
$QGJ_n+\overline{QGJ_n}=\left(QJ_n+iQJ_{n-1}\right)+\left(QJ_n-iQJ_{n-1}\right)=2QJ_n$.
5. The equation (18) is obtained from the equation (16).
6. From the Binet Formula of GJ_n, we get the desired relation.

In the following theorem we will give the identities for QGJ_n and QGj_n analogous to (5) and (6).

Theorem 2.3. Let QGJ_n be the Gaussian Jacobsthal quaternion and QGj_n be Gaussian Jacobsthal-Lucas quaternion. Then we have the following relations:

1. $QGJ_{n+1}+2QGJ_{n-1}=QGj_n$,
2. $2QGJ_{n+1}-QGJ_n=QGj_n$.

Proof.

1. From the relations $QGJ_n=QJ_n+iQJ_{n-1}$ and $QGj_n=Qj_n+iQj_{n-1}$ and the identity $QJ_{n+1}+2QJ_{n-1}=Qj_n$ given in Theorem 2.2 of [11], it follows that

$$\begin{aligned} QGJ_{n+1}+2QGJ_{n-1} &= \left(QJ_{n+1}+iQJ_n\right)+2\left(QJ_{n-1}+iQJ_{n-2}\right) \\ &= \left(QJ_{n+1}+2QJ_{n-1}\right)+i\left(QJ_n+2QJ_{n-2}\right) \\ &= Qj_n+iQj_{n-1}=QGj_n. \end{aligned}$$

2. From the relations $QGJ_n = QJ_n + iQJ_{n-1}$ and $QGj_n = Qj_n + iQj_{n-1}$ and the identity given in Theorem 3 of [19], we get the following identity analogous to (5)

$$\begin{aligned} 2QGJ_{n+1} - QGJ_n &= (2QJ_{n+1} + 2iQJ_n) - (QJ_n + iQJ_{n-1}) \\ &= (2QJ_{n+1} - QJ_n) + i(2QJ_n - QJ_{n-1}) \\ &= Qj_n + iQj_{n-1} = QGj_n. \end{aligned}$$

Proposition 2.4. The identities listed below hold for the Gaussian Jacobsthal quaternion QGJ_n and Gaussian Jacobsthal-Lucas quaternion QGj_n.

1. $QGJ_n + QGJ_{n+1} = 2^{n-1}(2+i)(1+2e_1+4e_2+8e_3)$,
2. $QGJ_{n+1} - QGJ_n = \frac{1}{3}\left[2^{n-1}(2+i)(1+2e_1+4e_2+8e_3) + 2(-1)^{n-1}(i-1)(1-e_1+e_2-e_3)\right]$,
3. $QGj_n + QGj_{n+1} = 3\cdot 2^{n-1}(2+i)(1+2e_1+4e_2+8e_3)$,
4. $QGj_{n+1} - QGj_n = 2^{n-1}(2+i)(1+2e_1+4e_2+8e_3) + 2(-1)^n(i-1)(1-e_1+e_2-e_3)$,
5. $3QGJ_n + QGj_n = 2^n(2+i)(1+2e_1+4e_2+8e_3)$.

Proof.

1. From the relation $QGJ_n = QJ_n + iQJ_{n-1}$, it follows that

$$\begin{aligned} QGJ_n + QGJ_{n+1} &= (QJ_n + iQJ_{n-1}) + (QJ_{n+1} + iQJ_n) \\ &= (QJ_n + QJ_{n+1}) + i(QJ_{n-1} + QJ_n) \\ &= 2^n(1+2e_1+4e_2+8e_3) + i2^{n-1}(1+2e_1+4e_2+8e_3) \\ &= 2^{n-1}(2+i)(1+2e_1+4e_2+8e_3). \end{aligned}$$

2. From the relation $QGJ_n = QJ_n + iQJ_{n-1}$, we get

$$\begin{aligned} QGJ_{n+1} - QGJ_n &= (QJ_{n+1} + iQJ_n) - (QJ_n + iQJ_{n-1}) \\ &= (QJ_{n+1} - QJ_n) + i(QJ_n - QJ_{n-1}) \\ &= \frac{1}{3}\left[2^n(1+2e_1+4e_2+8e_3) + 2(-1)^n(1-e_1+e_2-e_3)\right] + \\ &\quad \frac{i}{3}\left[2^{n-1}(1+2e_1+4e_2+8e_3) + 2(-1)^{n-1}(1-e_1+e_2-e_3)\right] \\ &= \frac{1}{3}\left[2^{n-1}(2+i)(1+2e_1+4e_2+8e_3) + 2(-1)^{n-1}(i-1)(1-e_1+e_2-e_3)\right]. \end{aligned}$$

3. Using the relation $QGj_n = Qj_n + iQj_{n-1}$, then we obtain

$$\begin{aligned} QGj_n + QGj_{n+1} &= (Qj_n + iQj_{n-1}) + (Qj_{n+1} + iQj_n) \\ &= (Qj_n + Qj_{n+1}) + i(Qj_{n-1} + Qj_n) \\ &= 3\cdot 2^n(1+2e_1+4e_2+8e_3) + i3\cdot 2^{n-1}(1+2e_1+4e_2+8e_3) \\ &= 3\cdot 2^{n-1}(2+i)(1+2e_1+4e_2+8e_3). \end{aligned}$$

4. If we take into account the relation $QGj_n = Qj_n + iQj_{n-1}$, then we have

$$\begin{aligned} QGj_{n+1} - QGj_n &= (Qj_{n+1} + iQj_n) - (Qj_n + iQj_{n-1}) \\ &= (Qj_{n+1} - Qj_n) + i(Qj_n - Qj_{n-1}) \\ &= \left[2^n(1+2e_1+4e_2+8e_3) + 2(-1)^{n+1}(1-e_1+e_2-e_3)\right] + \\ &\quad i\left[2^{n-1}(1+2e_1+4e_2+8e_3) + 2(-1)^n(1-e_1+e_2-e_3)\right] \\ &= 2^{n-1}(2+i)(1+2e_1+4e_2+8e_3) + 2(-1)^n(i-1)(1-e_1+e_2-e_3). \end{aligned}$$

5. When used similar arguments as above, we conclude

$$\begin{aligned}3QGJ_n + QGj_n &= (3QJ_n + 3iQJ_{n-1}) + (Qj_n + iQj_{n-1}) \\ &= (3QJ_n + Qj_n) + i(3QJ_{n-1} + Qj_{n-1}) \\ &= 2^{n+1}(1 + 2e_1 + 4e_2 + 8e_3) + i2^n(1 + 2e_1 + 4e_2 + 8e_3) \\ &= 2^n(2 + i)(1 + 2e_1 + 4e_2 + 8e_3).\end{aligned}$$

The next corollary follows from the relation $QGJ_n = QJ_n + iQJ_{n-1}$ and Theorem 2.4 of [11].

Corollary 2.5. Let QGJ_n be the n^{th} Gaussian Jacobsthal quaternion. Then we have the following summation formulas:

1. $\sum_{s=1}^{n} QGJ_s = \frac{1}{2}\left[QGJ_{n+2} - QGJ_2\right],$
2. $\sum_{s=1}^{n} QGJ_{2s-1} = \frac{2}{3}QGJ_{2n} + \frac{1}{3}\left[\left(n(1-i)+i\right)\left(2QJ_2 - QJ_3\right) + (3i-2)QJ_0 + 2iQJ_1\right]$
3. $\sum_{s=1}^{n} QGJ_{2s} = \frac{2}{3}QGJ_{2n+1} - \frac{1}{3}\left[n(1-i)\left(2QJ_2 - QJ_3\right) - 2QJ_1 + 2iQJ_0\right].$

Theorem 2.6. (Binet Formulas). Let QGJ_n be the Gaussian Jacobsthal quaternion and QGj_n be the Gaussian Jacobsthal-Lucas quaternion. The Binet formulas for these quaternions are

$$QGJ_n = \frac{1}{\alpha - \beta}\left[A\alpha^{n-1}(\alpha + i) - B\beta^{n-1}(\beta + i)\right]$$

and

$$QGj_n = 3A\alpha^{n-1}(\alpha + i) + 3B\beta^{n-1}(\beta + i)$$

respectively, where $\alpha = 2, \beta = -1,$ $A = 1 + 2e_1 + 4e_2 + 8e_3$ and $B = 1 - e_1 + e_2 - e_3.$

Proof. The characteristic equation of recurrence relations $QGJ_{n+1} = QGJ_n + 2QGJ_{n-1}$ and $QGj_{n+1} = QGj_n + 2QGj_{n-1}$ proved in Proposition 2.2 is $t^2 - t - 2 = 0.$ If we solve this quadratic equation, then we obtain the roots $\alpha = 2$ and $\beta = -1$.When we use the relation $QGJ_n = QJ_n + iQJ_{n-1}$ and the Binet Formula of real Jacobsthal quaternion QJ_n given in (11), then we get

$$\begin{aligned}QGJ_n &= QJ_n + iQJ_{n-1} \\ &= \frac{1}{\alpha - \beta}\left(A\alpha^n - B\beta^n\right) + i\frac{1}{\alpha - \beta}\left(A\alpha^{n-1} - B\beta^{n-1}\right) \\ &= \frac{1}{\alpha - \beta}\left[A\alpha^{n-1}(\alpha + i) - B\beta^{n-1}(\beta + i)\right].\end{aligned}$$

We can similarly derive the Binet Formula of QGj_n by means of the relation $QGj_n = Qj_n + iQj_{n-1}$ and the Binet Formula of real Jacobsthal-Lucas quaternion Qj_n given in (11) as follows:

$$\begin{aligned}QGj_n &= Qj_n + iQj_{n-1} \\ &= \left(3A\alpha^n + 3B\beta^n\right) + i\left(3A\alpha^{n-1} + 3B\beta^{n-1}\right) \\ &= 3A\alpha^{n-1}(\alpha + i) + 3B\beta^{n-1}(\beta + i).\end{aligned}$$

Theorem 2.7. (Cassini Identity). Let QGJ_n and QGj_n be n^{th} Gaussian Jacobsthal quaternion and Gaussian Jacobsthal-Lucas quaternion, respectively. For $n \geq 1$, the Cassini identities of QGJ_n and QGj_n are given in the following form:

$$QGJ_{n-1}QGJ_{n+1} - QGJ_n^{\ 2} = (-1)^{n-1} 2^{n-2} (7 + 5e_1 + 7e_2 + 5e_3)(i+2)(i-1)$$

$$QGj_{n-1}QGj_{n+1} - QGj_n^{\ 2} = (-1)^{n-2} 2^{n-2} 3^4 (7 + 5e_1 + 7e_2 + 5e_3)(i+2)(i-1).$$

Proof. In order to derive Cassini identity for Gaussian Jacobsthal quaternion QGJ_n, we will use its Binet Formula. Thus,

$$\begin{aligned}
QGJ_{n-1}QGJ_{n+1} - QGJ_n^{\ 2} &= \frac{1}{\alpha-\beta}\left[A\alpha^{n-2}(\alpha+i) - B\beta^{n-2}(\beta+i)\right]\cdot\frac{1}{\alpha-\beta}\left[A\alpha^{n}(\alpha+i) - B\beta^{n}(\beta+i)\right] \\
&\quad - \left(\frac{1}{\alpha-\beta}\left[A\alpha^{n-1}(\alpha+i) - B\beta^{n-1}(\beta+i)\right]\right)^2 \\
&= \frac{1}{9}\left[A^2\alpha^{2n-2}(i+2)^2 - AB\alpha^{n-2}\beta^{n}(i-1)(i+2) - BA\alpha^{n}\beta^{n-2}(i-1)(i+2) + B^2\beta^{2n-2}(i-1)^2\right] \\
&\quad - \frac{1}{9}\left[A^2\alpha^{2n-2}(i+2)^2 - AB\alpha^{n-1}\beta^{n-1}(i-1)(i+2) - BA\alpha^{n-1}\beta^{n-1}(i-1)(i+2) + B^2\beta^{2n-2}(i-1)^2\right] \\
&= \frac{1}{9}\left[AB\left((\alpha\beta)^{n-1} - (\alpha\beta)^{n-2}\right) + BA\left((\alpha\beta)^{n-1} - (\alpha\beta)^{n}\right)\right](i-1)(i+2) \\
&= \frac{(-1)^{n-1}2^{n-2}}{3}\left[AB + 2BA\right](i-1)(i+2) \\
&= (-1)^{n-1}2^{n-2}(7 + 5e_1 + 7e_2 + 5e_3)(i+2)(i-1)
\end{aligned}$$

where $AB = 7 - 11e_1 - e_2 + 13e_3$, $BA = 7 + 13e_1 + 11e_2 + e_3$ and $AB + 2BA = 21 + 15e_1 + 21e_2 + 15e_3$.

As for proving the Cassini identity of QGj_n, we use its Binet Formula. Therefore,

$$\begin{aligned}
QGj_{n-1}QGj_{n+1} - QGj_n^{\ 2} &= \left[3A\alpha^{n-2}(\alpha+i) + 3B\beta^{n-2}(\beta+i)\right]\cdot\left[3A\alpha^{n}(\alpha+i) + 3B\beta^{n}(\beta+i)\right] \\
&\quad - \left[3A\alpha^{n-1}(\alpha+i) + 3B\beta^{n-1}(\beta+i)\right]^2 \\
&= 9\left[A^2\alpha^{2n-2}(i+2)^2 + AB\alpha^{n-2}\beta^{n}(i-1)(i+2) + BA\alpha^{n}\beta^{n-2}(i-1)(i+2) + B^2\beta^{2n-2}(i-1)^2\right] \\
&\quad - 9\left[A^2\alpha^{2n-2}(i+2)^2 + AB\alpha^{n-1}\beta^{n-1}(i-1)(i+2) + BA\alpha^{n-1}\beta^{n-1}(i-1)(i+2) + B^2\beta^{2n-2}(i-1)^2\right] \\
&= 9\left[AB\left((\alpha\beta)^{n-2} - (\alpha\beta)^{n-1}\right) + BA\left((\alpha\beta)^{n} - (\alpha\beta)^{n-1}\right)\right](i-1)(i+2) \\
&= 9(-1)^{n-2}2^{n-2}\left[3AB + 6BA\right](i-1)(i+2) \\
&= (-1)^{n-2}2^{n-2}3^4(7 + 5e_1 + 7e_2 + 5e_3)(i+2)(i-1).
\end{aligned}$$

Proposition 2.8. Let n be a positive integer. Then we have the following summation formulas:

1) $\displaystyle\sum_{s=0}^{n}\binom{n}{s}2^{n-s}QGJ_s = QGJ_{2n}$,

2) $\displaystyle\sum_{s=0}^{n}\binom{n}{s}2^{n-s}QGj_s = QGj_{2n}$.

Proof. 1) From the Binet Formula of QGJ_n, we get

$$\sum_{s=0}^{n}\binom{n}{s}2^{n-s}QGJ_s=\sum_{s=0}^{n}\binom{n}{s}2^{n-s}\frac{1}{\alpha-\beta}\left[A\alpha^{s-1}(\alpha+i)-B\beta^{s-1}(\beta+i)\right]$$
$$=\frac{A(i+2)}{\alpha-\beta}\sum_{s=0}^{n}\binom{n}{s}2^{n-s}\alpha^{s-1}-\frac{B(i-1)}{\alpha-\beta}\sum_{s=0}^{n}\binom{n}{s}2^{n-s}\beta^{s-1}$$
$$=\frac{A(i+2)}{\alpha-\beta}\cdot\frac{1}{\alpha}\sum_{s=0}^{n}\binom{n}{s}2^{n-s}\alpha^{s}-\frac{B(i-1)}{\alpha-\beta}\cdot\frac{1}{\beta}\sum_{s=0}^{n}\binom{n}{s}2^{n-s}\beta^{s}$$
$$=\frac{A(i+2)}{\alpha-\beta}\cdot\frac{1}{\alpha}(\alpha+2)^n-\frac{B(i-1)}{\alpha-\beta}\cdot\frac{1}{\beta}(\beta+2)^n$$
$$=\frac{A(i+2)}{\alpha-\beta}\alpha^{2n-1}-\frac{B(i-1)}{\alpha-\beta}\beta^{2n-1}=QGJ_{2n}.$$

2) Using the Binet Formula of QGj_n, we conclude

$$\sum_{s=0}^{n}\binom{n}{s}2^{n-s}QGj_s=\sum_{s=0}^{n}\binom{n}{s}2^{n-s}\left[3A\alpha^{s-1}(\alpha+i)+3B\beta^{s-1}(\beta+i)\right]$$
$$=\frac{3A(i+2)}{\alpha}\sum_{s=0}^{n}\binom{n}{s}2^{n-s}\alpha^{s}+\frac{3B(i-1)}{\beta}\sum_{s=0}^{n}\binom{n}{s}2^{n-s}\beta^{s}$$
$$=\frac{3A(i+2)}{\alpha}(\alpha+2)^n+\frac{3B(i-1)}{\beta}(\beta+2)^n$$
$$=3A(i+2)\alpha^{2n-1}+3B(i-1)\beta^{2n-1}=QGj_{2n}.$$

Considering the equations (14) and (15), one can see Gaussian Jacobsthal and Gaussian Jacobsthal-Lucas quaternions satisfy a second-order linear recurrence relation. Thus we can derive the generating functions for these quaternions. Consequently, we give the following theorem.

Theorem 2.9. Let QGJ_n and QGj_n be n^{th} Gaussian Jacobsthal quaternion and Gaussian Jacobsthal-Lucas quaternion, respectively. The generating functions for QGJ_n and QGj_n are

$$g(x,t)=\frac{QGJ_0(1-t)+QGJ_1t}{1-t-2t^2}$$

and

$$h(x,t)=\frac{QGj_0(1-t)+QGj_1t}{1-t-2t^2}$$

where $QGJ_0=\frac{i}{2}+e_1+(1+i)e_2+(3+i)e_3$, $QGJ_1=1+(1+i)e_1+(3+i)e_2+(5+3i)e_3$, $QGj_0=2-\frac{i}{2}+(1+2i)e_1+(5+i)e_2+(7+5i)e_3$ and $QGj_1=1+2i+(5+i)e_1+(7+5i)e_2+(17+7i)e_3$ are initial values for these quaternions.

Proof. Let $g(x,t)=\sum_{n=0}^{\infty}QGJ_n(x)t^n$ be the generating function of QGJ_n. Then with simple calculations,

$$tg(x,t)=\sum_{n=0}^{\infty}QGJ_n(x)t^{n+1}=QGJ_0t+QGJ_1t^2+QGJ_2t^3+QGJ_3t^4+\cdots+QGJ_nt^{n+1}+\cdots$$

and

$$2t^2 g(x,t)=\sum_{n=0}^{\infty}2QGJ_n(x)t^{n+2}=2QGJ_0t^2+2QGJ_1t^3+2QGJ_2t^4+2QGJ_3t^5+\cdots+2QGJ_nt^{n+2}+\cdots$$

can be written. When we substract $tg(x,t)$ and $2t^2g(x,t)$ from $g(x,t)$, then we get

$$\left(1-t-2t^2\right)g(x,t)=QGJ_0+\left(QGJ_1-QGJ_0\right)t=QGJ_0\left(1-t\right)+QGJ_1t\ .$$

So we get desired.

$h(x,t)=\sum_{n=0}^{\infty}QGj_n(x)t^n$ be the generating function of QGj_n. Since the second-order linear recurrence relation of the Gaussian Jacobsthal-Lucas quaternions is the same as the Gaussian Jacobsthal-Lucas quaternions, then we have

$$th(x,t)=\sum_{n=0}^{\infty}QGj_n(x)t^{n+1}=QGj_0t+QGj_1t^2+QGj_2t^3+QGj_3t^4+\cdots+QGj_nt^{n+1}+\cdots$$

and

$$2t^2h(x,t)=\sum_{n=0}^{\infty}2QGj_n(x)t^{n+2}=2QGj_0t^2+2QGj_1t^3+2QGj_2t^4+2QGj_3t^5+\cdots+2QGj_nt^{n+2}+\cdots$$

Substracting $th(x,t)$ and $2t^2h(x,t)$ from $h(x,t)$, then we obtain

$$\left(1-t-2t^2\right)h(x,t)=QGj_0+\left(QGj_1-QGj_0\right)t=QGj_0\left(1-t\right)+QGj_1t\ .$$

References

[1] Hamilton, W. R., Elements of Quaternions, Longmans, Gren and Co., London, 1866.

[2] Chou, J. C. K., Quaternion Kinematics and Dynamic Differantial Equation, *IEEE Transaction on Robotics and Automation*, 8(1) (1992) 53-63.

[3] Conte, E., Biquaternion Quantum Mechanics, Pitagora Editrice, Via del Legatore, Blogna, Italy, 2000.

[4] Conway, A. W., The Quaternionic Form of Relativity, *Phil. Mag*., 24 (1912) 208-211.

[5] Gurlebeck, K., Sprossig, W., Quaternionic and Clifford Calculus for Physicists and Engineers, John Wiley & Sons, Chichester, New York, 1997.

[6] Jolly, D. C., Isomorphism between matrices and quaternions, *Lett. Nuovo Cimento*., 44(2) (1985) 80-82.

[7] Negi, O. P. S., Bisht, S., Bisht, P. S., Revisiting Quaternion Formulation and Electromagnetism, *I1 Nuovo Cimento*, 113B(12) (1998) 1449-1467.

[8] Tanişli, M., Özdaş, K., Application of Quaternion Representation to Stanford Manipulator, *Balkan Physics Letters*, 5(2) (1997) 65-68.

[9] Horadam, A. F., Jacobsthal and Pell Curves, *Fibonacci Quart*., 26 (1988) 79-83.

[10] Horadam, A. F., Jacobsthal Representation Numbers, *Fibonacci Quart*., 34 (1996) 40-54.

[11] Aydın, Torunbalcı F., Yüce, S., A New Approach to Jacobsthal Quaternions, *Filomat*, 31(18) (2017) 5567-5579.

[12] Aydın, Torunbalcı F., On Generalizations of the Jacobsthal Sequence, *Notes on Number Theory and Discrete Mathematics*, 24(1) (2018) 120-135.

[13] Catarino, P., The Modified Pell and Modified k-Pell Quaternions and Octonions, *Adv. Appl. Clifford Algebras*, 26(2) (2016) 577-590.

[14] Çimen, C. B., İpek, A., On Pell quaternions and Pell-Lucas quaternions, *Adv. Appl. Clifford Algebras*,

26(1) (2016) 39-51.

[15] Halici, S., On Fibonacci Quaternions, *Adv. Appl. Clifford Algebras*, 22(2) (2012) 321-327.

[16] Halici, S., On complex Fibonacci Quaternions, *Adv. Appl. Clifford Algebras*, 23 (2013) 105-112.

[17] Horadam, A. F., Complex Fibonacci Numbers and Fibonacci Quaternions, *American Math. Monthly*, 70(3) (1963) 289-291.

[18] Horadam, A. F., Quaternion recurrence relations, *Ulam Quart*., 2(2) (1993) 23-33.

[19] Szynal-Liana, A., Wloch, I., A Note on Jacobsthal Quaternions, *Adv. Appl. Clifford Algebr*., 26(1) (2016) 441-447.

[20] Aşçı, M., Gürel, E., Gaussian Jacobsthal and Gaussian Jacobsthal-Lucas polynomials, *Note on Number Theory and Discrete Mathematics*, 19(1) (2013) 25-36.

[21] Aşçı, M., Gürel, E., Gaussian Jacobsthal and Gaussian Jacobsthal-Lucas numbers, *Ars Combin.*, 111 (2013) 53-63.

[22] Cerin, Z., Formulae for Sums of Jacobsthal-Lucas Numbers, *International Mathematical Forum*, 2(40) (2007) 1969–1984.

[23] Daşdemir, A., On the Jacobsthal Numbers by Matrix Method, *SDU Journal of Science (E-Journal)*, 71 (2012) 69–76.

[24] Koshy, T., Fibonacci and Lucas Numbers with Applications, A Wiley-Interscience Publication, 2001.

[25] Köken, F., Bozkurt, D., On the Jacobsthal Numbers by Matrix Methods, *International Journal of Contemporary Mathematical Sciences*, 3(13) (2008) 605–614.

[26] Jordan, J. H., Gaussian Fibonacci and Lucas Numbers, *Fibonacci Quart*., 3 (1965) 315-318.

[27] Pethe, S., Horadam, A. F., Generalized Gaussian Fibonacci Numbers, *Bull. Austral. Math. Soc*., 33(1) (1986) 37-48.

[28] Stephan, J. S., Fundamental Representations and Algebraic Properties of Biquaternions or Complexified Quaternions, *Adv. Appl. Clifford Algebras*, 21 (2011) 607-636.