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On some geometric properties of normalized Wright functions

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Abstract

The main purpose of the present paper is to determine the radii of lemniscate starlikeness, lemniscate convexity, Janowski starlikeness and Janowski convexity of normalized Wright functions. The key tools in the proof of our main results are the infinite product representation of Wright function and some properties of real zeros of Wright function and its derivative.

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1. Introduction and Main Results

By $\mathbb{D}_r$ we mean the open disk $\{ z \in \mathbb{C}: |z| < r \}$ with the radius $r > 0$ and let $\mathbb{D} = \mathbb{D}_1$. Let $f: \mathbb{D}_r \to \mathbb{C}$ be the function defined by

$$f(z) = z + \sum_{n\geq 2} a_n z^n \tag{1}$$

where r is less or equal than the radius of convergence of the above power series. Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form (1) which are analytic and normalized by the conditions $f(0) = f'(0) - 1 = 0$ in the open disk $\mathbb{D}_r$. By $\mathcal{S}$ we mean the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{D}_r$. A function $f \in \mathcal{A}$ is said to be starlike function if $f(\mathbb{D}_r)$ is starlike domain with the respect to the origin, meaning that for each $z \in \mathbb{D}_r$ the segment between the origin and $f(z)$ lie in $f(\mathbb{D}_r)$. Moreover, the function f, defined by (1), is said to be convex function in $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and the image domain $f(\mathbb{D}_r)$ is a convex domain, meaning that it is starlike domain with respect to each of its points. A function $f \in \mathcal{A}$ is subordinate to a function $g \in \mathcal{A}$, written as $f(z) \prec g(z)$, if there exist a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In addition, we know that if g is a univalent function, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. It is clear that various subclasses of starlike and convex functions can be unified by using the concept of subordination. Let φ be analytic function, then by $S^\star(\varphi)$ we mean the class of all analytic functions satisfying $1 + \frac{zf'(z)}{f(z)} \prec \varphi(z)$. Moreover, we denote the class of all analytic functions satisfying $1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z)$ by $\mathcal{K}(\varphi)$. It is worth mentioning that these classes give a unified presentation of several famous subclasses of starlike and convex functions. For instance, if we take $\varphi(z) = \sqrt{1+z}$, we get the classes of lemniscate starlike and lemniscate convex functions denoted by $S^\star_{\mathcal{L}} := S^\star\left(\sqrt{1+z}\right)$ and $\mathcal{K}_{\mathcal{L}} := \mathcal{K}\left(\sqrt{1+z}\right)$, respectively. Lemniscate starlike functions introduced and investigated by Sokól and Stankiewich [15].

We denote the radius of lemniscate starlikeness of function f by the real number

$$r^\star_{\mathcal{L}}(f) = \sup\left\{ r > 0 \middle| \left| \left(\frac{zf'(z)}{f(z)}\right)^2 - 1 \right| < 1 \text{ for } z \in \mathbb{D}_r, 0 \leq r \leq 1 \right\}.$$

In addition, the radius of lemniscate convexity of function f is given by the real number

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$$r_{\mathcal{L}}^{c}(f)=\sup\left\{r>0\left|\ \left|\left(\frac{zf''(z)}{f'(z)}\right)^{2}-1\right|<1 \text{ for } \mathrm{z}\in\mathbb{D}_{\mathrm{r}}, 0\leq r\leq 1\right.\right\}.$$

If we take $\varphi(z)=\frac{1+Az}{1+Bz}$ for $-1\leq\ B<A\leq 1$, we have the classes of Janowski starlike and Janowski convex functions denoted by $S^{\star}[A,B]:=S^{\star}(\frac{1+Az}{1+Bz})$ and $\mathcal{K}[A,B]:=\mathcal{K}\left(\frac{1+Az}{1+Bz}\right)$, respectively. Recall here that Janowski function is defined in the form $\frac{1+Az}{1+Bz}$ for $-1\leq B<A\leq 1$. For comprehensive knowledge on Janowski starlike and Janowski convex functions one may refer to [10].

The real number

$$r_{A,B}^{\star}(f)\ =\sup\left\{r>0\left|\ \left|\frac{\left(\frac{zf'(z)}{f(z)}\right)-1}{A-B\left(\frac{zf'(z)}{f(z)}\right)}\right|<1 \text{ for } \mathrm{z}\in\mathbb{D}_{\mathrm{r}}, 0\leq r\leq 1\right.\right\}$$

is called as the radius of Janowski starlikeness of the function f.

Moreover, the real number

$$r_{A,B}^{c}(f)=\sup\left\{r>0\left|\ \left|\frac{\frac{zf''(z)}{f'(z)}}{A-B\left(1+\frac{zf''(z)}{f'(z)}\right)}\right|<1\ \text{ for } \mathrm{z}\in\mathbb{D}_{\mathrm{r}}, 0\leq r\leq 1\right.\right\}$$

is called as the radius of Janowski convexity of the function f.

Because of the fact that special functions play a key role in solving many problems in mathematics, applied mathematics and physics they have received a great deal of attention by many researchers from geometric function theory field (see the recent papers[1-5,7,9,17] and the references therein). In addition, for comprehensive knowledge on the radius problems of some special functions, one may consult on the works [6,10,13,16,18]. Recently, in [14] the authors investigated some geometric properties such that the radii of lemniscate starlikeness and lemniscat convexity, Janowski starlikeness and Janowski convexity of Bessel functions.

Inspired by the above mentioned papers, our main goal in the present paper is to ascertain the radii of starlikeness and convexity associated with lemniscate of Bernoulli and the Janowski function of normalized Wright functions.

Let us take a look the Wright function represented by

$$\phi(\rho,\beta,z)=\sum_{n\geq 0}\frac{z^{n}}{n!\,\Gamma(n\rho+\beta)},$$

where $\rho>-1$ and $z,\beta\in\mathbb{C}$. For further knowledge on Wright function one may refer to the studies [11] and [19]. Furthermore, it is worth mentioning here that Wright function is an entire function of z for $\rho>-1$. This means that some properties of the entire functions can be also used for Wright function.

With the aid of [9] we know that if $\rho>0$ and $\beta>0$, then the function $z\mapsto\lambda_{\rho,\beta}(z)=\phi(\rho,\beta,-z^{2})$ has infinitely many zeros which are all real. If we denote the nth positive zero of the function $z\mapsto\phi(\rho,\beta,-z^{2})$ by $\lambda_{\rho,\beta,n}$, under the same conditions the infinite product representation

$$\Gamma(\beta)\phi(\rho,\beta,-z^{2})=\prod_{n\geq 1}\left(1-\frac{z^{2}}{\lambda_{\rho,\beta,n}^{2}}\right) \tag{2}$$

holds, and this product is uniformly convergent on compact subsets of the complex plane. Moreover, let $\zeta_{\rho,\beta,n}'$ be the nth positive zero of $\Psi_{\rho,\beta}'$, where $\Psi_{\rho,\beta}(z)=z^{\beta}\lambda_{\rho,\beta}(z)$, then between any two positive zeros of $\lambda_{\rho,\beta}$ (or the positive real zeros of the function $\Psi_{\rho,\beta}$) there must be precisely one of $\Psi_{\rho,\beta}'$. That is, the zeros satisfy the following inequalities

$$\zeta_{\rho,\beta,1}'<\lambda_{\rho,\beta,1}<\zeta_{\rho,\beta,2}'<\lambda_{\rho,\beta,2}<\cdots.$$

We can readily observe that the function $z\mapsto\phi(\rho,\beta,-z^{2})$ is not included in the class $\mathcal{A}$, and thus first we perform some natural normalization. Now we deal with three functions deriving from $\phi(\rho,\beta,\cdot)$:

$$f_{\rho,\beta}(z) = \left(z^{\beta}\Gamma(\beta)\phi(\rho,\beta,-z^2)\right)^{\frac{1}{\beta}}, \tag{3}$$
$$g_{\rho,\beta}(z) = z\Gamma(\beta)\phi(\rho,\beta,-z^2), \tag{4}$$
$$h_{\rho,\beta}(z) = z\Gamma(\beta)\phi(\rho,\beta,-z)\,. \tag{5}$$

Clearly these functions are in the class $\mathcal{A}$. There is no doubt it is possible to obtain infinitely many other normalizations, that is the reason why we focus on the above mentioned ones is the fact that we oftenly encounter with their particular cases in terms of Bessel functions in the literature.

1.1. Lemniscate starlikeness and lemniscate convexity of normalized Wright functions

In the present section we deal with the radii of starlikeness and convexity associated with lemniscate of Bernoulli of the normalized Wright functions. It is important to emphasize here that it is said to be lemniscate starlike of a normalized analytic function f if the quantity $\frac{zf'(z)}{f(z)}$ lies in the region bounded by the right half of the lemniscate of Bernoulli $|w^2-1|=1$.

Theorem 1. Let $\rho>0$ and $\beta>0$. Then the following assertions are valid.

a. The radius of lemniscate starlikeness $r_{\mathcal{L}}^{\star}(f_{\rho,\beta})$ is the smallest positive root of the following transcendental equation
$$r^2\left(\lambda'_{\rho,\beta}(r)\right)^2 - 2\beta\lambda'_{\rho,\beta}(r)\lambda_{\rho,\beta}(r) - \beta^2\left(\lambda_{\rho,\beta}(r)\right)^2 = 0.$$

b. The radius of lemniscate starlikeness $r_{\mathcal{L}}^{\star}(g_{\rho,\beta})$ is the smallest positive root of the following transcendental equation
$$r^2\left(\lambda'_{\rho,\beta}(r)\right)^2 - 2r\lambda'_{\rho,\beta}(r)\lambda_{\rho,\beta}(r) - \left(\lambda_{\rho,\beta}(r)\right)^2 = 0.$$

c. The radius of lemniscate starlikeness $r_{\mathcal{L}}^{\star}(h_{\rho,\beta})$ is the smallest positive root of the following transcendental equation
$$r\left(\lambda'_{\rho,\beta}(\sqrt{r})\right)^2 - 4\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r})\lambda_{\rho,\beta}(\sqrt{r}) - \left(\lambda_{\rho,\beta}(\sqrt{r})\right)^2 = 0.$$

Proof. Owing to the Weierstrassian canonical representation of the function $z \mapsto \lambda_{\rho,\beta}(z)$ given in Eq. (2) we get
$$\frac{z\lambda'_{\rho,\beta}(z)}{\lambda_{\rho,\beta}(z)} = -\sum_{n\geq 1}\frac{2z^2}{\lambda^2_{\rho,\beta,n}-z^2}. \tag{6}$$

Bearing in mind the normalizations presented in Eqs. (3)-(5) and by making use of Eq. (6) we obtain the following equations
$$\frac{zf'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)} = 1+\frac{1}{\beta}\left(\frac{z\lambda'_{\rho,\beta}(z)}{\lambda_{\rho,\beta}(z)}\right) = 1-\frac{1}{\beta}\sum_{n\geq 1}\frac{2z^2}{\lambda^2_{\rho,\beta,n}-z^2}, \tag{7}$$

$$\frac{zg'_{\rho,\beta}(z)}{g_{\rho,\beta}(z)} = 1+\frac{z\lambda'_{\rho,\beta}(z)}{\lambda_{\rho,\beta}(z)} = 1-\sum_{n\geq 1}\frac{2z^2}{\lambda^2_{\rho,\beta,n}-z^2}\,, \tag{8}$$

$$\frac{zh'_{\rho,\beta}(z)}{h_{\rho,\beta}(z)} = 1+\frac{1}{2}\left(\frac{\sqrt{z}\lambda'_{\rho,\beta}(\sqrt{z})}{\lambda_{\rho,\beta}(\sqrt{z})}\right) = 1-\sum_{n\geq 1}\frac{z}{\lambda^2_{\rho,\beta,n}-z}. \tag{9}$$

By means of Eqs. (7), (8) and (9), we deduce that the following inequalities
$$\left|\left(\frac{z\,f'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)}\right)^2-1\right| \leq \frac{1}{\beta^2}\left(\sum_{n\geq 1}\frac{2|z|^2}{\lambda^2_{\rho,\beta,n}-|z|^2}\right)\left(\sum_{n\geq 1}\frac{2|z|^2}{\lambda^2_{\rho,\beta,n}-|z|^2}+2\beta\right)$$
$$= \left(\frac{|z|f'_{\rho,\beta}(|z|)}{f_{\rho,\beta}(|z|)}\right)^2 - 4\left(\frac{|z|f'_{\rho,\beta}(|z|)}{f_{\rho,\beta}(|z|)}\right)+3,$$

$$\left|\left(\frac{zg'_{\rho,\beta}(z)}{g_{\rho,\beta}(z)}\right)^2-1\right|\leq\left(\sum_{n\geq1}\frac{2|z|^2}{\lambda_{\rho,\beta,n}^2-|z|^2}\right)\left(\sum_{n\geq1}\frac{2|\,z|^2}{\lambda_{\rho,\beta,n}^2-|z|^2}+2\right)$$
$$=\left(\frac{|\,z|g'_{\rho,\beta}(|z|)}{g_{\rho,\beta}(|z|)}\right)^2-4\left(\frac{|z|g'_{\rho,\beta}(|z|\,)}{g_{\rho,\beta}(|z|)}\right)+3,$$
$$\left|\left(\frac{zh'_{\rho,\beta}(z)}{h_{\rho,\beta}(z)}\right)^2-1\right|\leq\left(\sum_{n\geq1}\frac{|z|}{\lambda_{\rho,\beta,n}^2-|z|}\right)\left(\sum_{n\geq1}\frac{|z|}{\lambda_{\rho,\beta,n}^2-|z|}+2\right)$$
$$=\left(\frac{|\,z|h'_{\rho,\beta}(|z|)}{h_{\rho,\beta}(|z|)}\right)^2-4\left(\frac{|z|h'_{\rho,\beta}(|z|\,)}{h_{\rho,\beta}(|z|)}\right)+3,$$

hold true for $|z|<\lambda_{\rho,\beta,1}$, $\rho>0$ and $\beta>0$.

For convenience, we deal with the function $z\mapsto\phi_{\rho,\beta}$ which collectively corresponds the functions $f_{\rho,\beta}, g_{\rho,\beta}$ and $h_{\rho,\beta}$. Assume that $r^\star$ is the smallest positive root of the equation

$$\left(\frac{r\phi'_{\rho,\beta}(r)}{\phi_{\rho,\beta}(r)}\right)^2-4\left(\frac{r\phi'_{\rho,\beta}(r)}{\phi_{\rho,\beta}(r)}\right)+2=0\,,$$

then the inequality

$$\left|\left(\frac{r\phi'_{\rho,\beta}(r)}{\phi_{\rho,\beta}(r)}\right)^2-1\right|\leq1$$

is valid for $|z|<r^\star$. With the help of Eqs. (7)-(9), it is deducible that the zeros of the above mentioned equation for the functions $f_{\rho,\beta}, g_{\rho,\beta}$ and $h_{\rho,\beta}$ coincide with those of equations, respectively

$$r^2\left(\lambda'_{\rho,\beta}(r)\right)^2-2\beta\lambda'_{\rho,\beta}(r)\lambda_{\rho,\beta}(r)-\beta^2\left(\lambda_{\rho,\beta}(r)\right)^2=0,$$
$$r^2\left(\lambda'_{\rho,\beta}(r)\right)^2-2r\lambda'_{\rho,\beta}(r)\lambda_{\rho,\beta}(r)-\left(\lambda_{\rho,\beta}(r)\right)^2=0$$

and

$$r\left(\lambda'_{\rho,\beta}(\sqrt{r})\right)^2-4\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r})\lambda_{\rho,\beta}(\sqrt{r})-\left(\lambda_{\rho,\beta}(\sqrt{r})\right)^2=0.$$

This means that the radii of lemniscate starlikeness $r^\star_{\mathcal{L}}(f_{\rho,\beta}), r^\star_{\mathcal{L}}(g_{\rho,\beta})$ and $r^\star_{\mathcal{L}}(h_{\rho,\beta})$ are the smallest positive roots of the above equations, respectively.

To complete the proof we must show that each of the above equations have a unique roots in the open interval $(0,\lambda_{\rho,\beta,1})$. To do this, let us take into consideration the functions $F_{\rho,\beta}, G_{\rho,\beta}, H_{\rho,\beta}\colon\left(0,\lambda_{\rho,\beta,1}\right)\to\mathbb{R}$ defined by

$$F_{\rho,\beta}(r)=\left(\frac{rf'_{\rho,\beta}(r)}{f_{\rho,\beta}(r)}\right)^2-4\left(\frac{rf'_{\rho,\beta}(r)}{f_{\rho,\beta}(r)}\right)+2,$$
$$G_{\rho,\beta}(r)=\left(\frac{rg'_{\rho,\beta}(r)}{g_{\rho,\beta}(r)}\right)^2-4\left(\frac{rg'_{\rho,\beta}(r)}{g_{\rho,\beta}(r)}\right)+2,$$
$$H_{\rho,\beta}(r)=\left(\frac{rh'_{\rho,\beta}(r)}{h_{\rho,\beta}(r)}\right)^2-4\left(\frac{rh'_{\rho,\beta}(r)}{h_{\rho,\beta}(r)}\right)+2.$$

It is obvious that the above mentioned functions are strictly increasing functions of r since

$$F'_{\rho,\beta}(r)=\frac{2}{\beta}\sum_{n\geq1}\frac{4r\lambda_{\rho,\beta,n}^2}{\left(\lambda_{\rho,\beta,n}^2-r^2\,\right)^2}\left(1+\frac{1}{\beta}\sum_{n\geq1}\frac{2r^2}{\lambda_{\rho,\beta,n}^2-r^2}\right)>0,$$
$$G'_{\rho,\beta}(r)=\sum_{n\geq1}\frac{4r\lambda_{\rho,\beta,n}^2}{\left(\lambda_{\rho,\beta,n}^2-r^2\right)^2}\left(1+\sum_{n\geq1}\frac{2r^2}{\lambda_{\rho,\beta,n}^2-r^2}\right)>0,$$

$$H'_{\rho,\beta}(r) = 2\sum_{n\geq 1}\frac{\lambda^2_{\rho,\beta,n}}{\left(\lambda^2_{\rho,\beta,n}-r\right)^2}\left(1+\sum_{n\geq 1}\frac{r}{\lambda^2_{\rho,\beta,n}-r}\right) > 0.$$

We can readily evaluate the limits

$$\lim_{r\searrow 0} F_{\rho,\beta}(r) = \lim_{r\searrow 0} G_{\rho,\beta}(r) = \lim_{r\searrow 0} H_{\rho,\beta}(r) = -1 < 0$$

and

$$\lim_{r\nearrow \lambda_{\rho,\beta,1}} F_{\rho,\beta}(r) = \lim_{r\nearrow \lambda_{\rho,\beta,1}} G_{\rho,\beta}(r) = \lim_{r\nearrow \lambda_{\rho,\beta,1}} H_{\rho,\beta}(r) = \infty.$$

This means that the lemniscate starlikeness radii of the functions $f_{\rho,\beta}, g_{\rho,\beta}$ and $h_{\rho,\beta}$, denoted by $r^{\star}_{\mathcal{L}}(f_{\rho,\beta}), r^{\star}_{\mathcal{L}}(g_{\rho,\beta})$ and $r^{\star}_{\mathcal{L}}(h_{\rho,\beta})$ are, respectively, the unique zeros of $F_{\rho,\beta}(r)$, $G_{\rho,\beta}(r)$ and $H_{\rho,\beta}(r)$ in the open interval $(0,\lambda_{\rho,\beta,1})$.

Theorem 2. Let $\rho > 0$ and $\beta > 0$. Then the following assertions are valid.

a. The radius of lemniscate convexity $r^{c}_{\mathcal{L}}(f_{\rho,\beta})$ is the smallest positive root of the following transcendental equation

$$\left(\frac{z\Psi''_{\rho,\beta}(z)}{\Psi'_{\rho,\beta}(z)}+\left(\frac{1}{\beta}-1\right)\frac{z\Psi'_{\rho,\beta}(z)}{\Psi_{\rho,\beta}(z)}\right)^2 - 2\left(\frac{z\Psi''_{\rho,\beta}(z)}{\Psi'_{\rho,\beta}(z)}+\left(\frac{1}{\beta}-1\right)\frac{z\Psi'_{\rho,\beta}(z)}{\Psi_{\rho,\beta}(z)}\right) - 1 = 0.$$

b. The radius of lemniscate convexity $r^{c}_{\mathcal{L}}(g_{\rho,\beta})$ is the smallest positive root of the following transcendental equation

$$\left(\frac{r^2\lambda''_{\rho,\beta}(r)+2r\lambda'_{\rho,\beta}(r)}{\lambda_{\rho,\beta}(r)+r\lambda'_{\rho,\beta}(r)}\right)^2 - 2\left(\frac{r^2\lambda''_{\rho,\beta}(r)+2r\lambda'_{\rho,\beta}(r)}{\lambda_{\rho,\beta}(r)+r\lambda'_{\rho,\beta}(r)}\right) - 1 = 0.$$

c. The radius of lemniscate convexity $r^{c}_{\mathcal{L}}(h_{\rho,\beta})$ is the smallest positive root of the following transcendental equation

$$\left(\frac{3\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r})-r\lambda''_{\rho,\beta}(\sqrt{r})}{4\lambda_{\rho,\beta}(\sqrt{r})+2\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r})}\right)^2 - 2\left(\frac{3\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r})-r\lambda''_{\rho,\beta}(\sqrt{r})}{4\lambda_{\rho,\beta}(\sqrt{r})+2\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r})}\right) - 1 = 0.$$

Proof

a. It is easy to obtain the following equality (see [9, Theorem 5])

$$1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)} = 1+\frac{z\Psi''_{\rho,\beta}(z)}{\Psi'_{\rho,\beta}(z)}+\left(\frac{1}{\beta}-1\right)\frac{z\Psi'_{\rho,\beta}(z)}{\Psi_{\rho,\beta}(z)}.$$

In view of [9] the infinite product representations

$$\Psi_{\rho,\beta}(z) = \frac{z^{\beta}}{\Gamma(\beta)}\prod_{n\geq 1}\left(1-\frac{z^2}{\zeta^2_{\rho,\beta,n}}\right) \text{ and } \Psi'_{\rho,\beta}(z) = \frac{z^{\beta-1}}{\Gamma(\beta)}\prod_{n\geq 1}\left(1-\frac{z^2}{\zeta'^{\,2}_{\rho,\beta,n}}\right)$$

are valid, where $\zeta_{\rho,\beta,n}$ and $\zeta'_{\rho,\beta,n}$ are the nth positive roots of $\Psi_{\rho,\beta}$ and $\Psi'_{\rho,\beta}$, respectively. By taking the logarithmic derivative on both sides of the above equalities we get

$$1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)} = 1-\sum_{n\geq 1}\frac{2z^2}{\zeta'^{\,2}_{\rho,\beta,n}-z^2}-\left(\frac{1}{\beta}-1\right)\sum_{n\geq 1}\frac{2z^2}{\zeta^2_{\rho,\beta,n}-z^2}. \tag{10}$$

Assume that $\beta\in(0,1]$, then $\frac{1}{\nu}-1\geq 0$. With the help of Eq. (10) and triangle inequality for $|z| < \zeta'_{\rho,\beta,1} < \zeta_{\rho,\beta,1}$, we arrive at

$$\left|\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right)^2-1\right| \leq \left(\sum_{n\geq 1}\frac{2|z|^2}{\zeta'^{\,2}_{\rho,\beta,n}-|z|^2}+\left(\frac{1}{\beta}-1\right)\sum_{n\geq 1}\frac{2|z|^2}{\zeta^2_{\rho,\beta,n}-|z|^2}\right)^2$$

$$+2\left(\sum_{n\geq 1}\frac{2|\,z|^2}{\zeta'^{\,2}_{\rho,\beta,n}-|\,z|^2}+\left(\frac{1}{\beta}-1\right)\sum_{n\geq 1}\frac{2|z|^2}{\zeta^2_{\rho,\beta,n}-|z|^2}\right).$$

By virtue of Eq. (10), the above inequality gives

$$\left|\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right)^2-1\right|\leq\left(\frac{|z|f''_{\rho,\beta}(|z|)}{f'_{\rho,\beta}(|z|)}\right)^2-2\left(\frac{|z|f''_{\rho,\beta}(|z|)}{f'_{\rho,\beta}(|z|)}\right). \tag{11}$$

Moreover, owing to the relation [8, Lemma 2.1]

$$\left|\frac{z}{a-z}-\lambda\frac{z}{b-z}\right|\leq\frac{|z|}{a-|z|}-\lambda\frac{|z|}{b-|z|} \tag{12}$$

for $|z|\leq r<a<b$ and $0\leq\lambda<1$ we conclude that the inequality given in Eq. (11) are valid for the case when $\beta>1$ as well. This means that the inequality obtained in Eq. (11) is valid for $\beta>0$ and $|z|<\zeta'_{\rho,\beta,1}$. Therefore, the function $f_{\rho,\beta}(z)$ lemniscate convex for $|z|<r_1$, where r_1 is the smallest positive root of

$$\left(\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}\right)^2-2\left(\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}\right)-1=0. \tag{13}$$

In order to complete the proof we must show that the above equation has a unique root in the open interval $(0,\zeta'_{\rho,\beta,1})$. To achieve our goal, let us take into account the function $u_{\rho,\beta}\colon(0,\zeta'_{\rho,\beta,1})\to\mathbb{R}$ defined by

$$u_{\rho,\beta}(r)=\left(\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}\right)^2-2\left(\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}\right)-1.$$

It is easy to see that this function is strictly increasing for $\beta>0$ since

$$u'_{\rho,\beta}(r)>\left(8r\sum_{n\geq 1}\left(\frac{\zeta'^2_{\rho,\beta,n}}{\left(\zeta^{('2)}_{\rho,\beta,n}-r^2\right)^2}-\frac{\zeta^2_{\rho,\beta,n}}{\left(\zeta^2_{\rho,\beta,n}-r^2\right)^2}\right)\right)\times$$
$$\left(2r^2\sum_{n\geq 1}\left(\frac{1}{\zeta'^{\,2}_{\rho,\beta,n}-r^2}-\frac{1}{\zeta^2_{\rho,\beta,n}-r^2}\right)\right)>0$$

Here we tacitly used the relation $\zeta^2_{\rho,\beta,n}\left(\zeta'^{\,2}_{\rho,\beta,n}-r^2\right)^2<\zeta'^{\,2}_{\rho,\beta,n}\left(\zeta^2_{\rho,\beta,n}-r^2\right)^2$ for $r<\sqrt{\zeta_{\rho,\beta,n}\zeta'_{\rho,\beta,n}}$ and $\beta>0$. Also it is simple to evaluate the limits

$$\lim_{r\searrow 0}u_{\rho,\beta}(r)=-1<0 \text{ and } \lim_{r\nearrow\zeta'_{\rho,\beta,1}}u_{\rho,\beta}(r)=\infty.$$

In light of all explanations above mentioned, it is deducible that the root is unique in the open interval $(0,\zeta'_{\rho,\beta,1})$ and the radius of lemniscate convexity of the function $f_{\rho,\beta}(z)$, represented by $r^c_{\mathcal{L}}(f_{\rho,\beta})$, is the unique root of Eq. (13) in the open interval $(0,\zeta'_{\rho,\beta,1})$.

b. We now focus on determining the radii of lemniscate convexity of normalized Wright function $g_{\rho,\beta}(z)$. Taking into account [9, Theorem 5], the infinite product representation of the function $g'_{\rho,\beta}(z)$ can be stated as

$$g'_{\rho,\beta}(z)=\prod_{n\geq 1}\left(1-\frac{z^2}{\vartheta^2_{\rho,\beta,n}}\right),$$

where $\vartheta_{\rho,\beta,n}$ is the nth positive zero of the function $g'_{\rho,\beta}(z)$. By taking the logarithmic derivative of the above equality we get

$$1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}=1-\sum_{n\geq 1}\frac{2z^2}{\vartheta^2_{\rho,\beta,n}-z^2}.$$

With the similar approach of the proof of Theorem 1, for $|\,z|<\vartheta_{\rho,\beta,1}$ we get

$$\left|\left(1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right)^2-1\right| \leq \left(\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right)^2-2\left(\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right).$$

Therefore, it follows that the radius of lemniscate convexity $r_{\mathcal{L}}^{c}(g_{\rho,\beta})$ is the unique positive root of the equation

$$\left(\frac{rg''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)}\right)^2-2\left(\frac{rg''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)}\right)-1=0 \tag{14}$$

in $(0,\vartheta_{\rho,\beta,1})$. To complete the proof we must show that the above equation has a unique root in the open interval $(0,\vartheta_{\rho,\beta,1})$. To achieve our goal, let us take into consideration the function $v_{\rho,\beta}:\left(0,\vartheta_{\rho,\beta,1}\right)\to\mathbb{R}$ defined by

$$v_{\rho,\beta}(r)=\left(\frac{rg''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)}\right)^2-2\left(\frac{rg''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)}\right)-1$$

which is strictly increasing in the same interval since

$$v'_{\rho,\beta}(r)=8r\left(\sum_{n\geq1}\frac{\vartheta_{\rho,\beta,n}^2}{\left(\vartheta_{\rho,\beta,n}^2-r^2\right)^2}\right)\left(\sum_{n\geq1}\frac{2r^2}{\left(\vartheta_{\rho,\beta,n}^2-r^2\right)^2}+1\right)>0.$$

Also it can be observed that

$$\lim_{r\searrow0}v_{\rho,\beta}(r)=-1<0 \text{ and } \lim_{r\nearrow\vartheta_{\rho,\beta,1}}v_{\rho,\beta}(r)=\infty.$$

That means that the radius of lemniscate convexity $r_{\mathcal{L}}^{c}(g_{\rho,\beta})$ is the unique root of Eq. (14) in the open interval $(0,\vartheta_{\rho,\beta,1})$.

c. In [9] it is shown that the infinite product representation of the function $h'_{\rho,\beta}(z)$ can be stated as

$$h'_{\rho,\beta}(z)=\prod_{n\geq1}\left(1-\frac{z}{\tau_{\rho,\beta,n}}\right),$$

where $\tau_{\rho,\beta,n}$ is the nth positive zero of the function $h'_{\rho,\beta}$. As a result of mimicking the same evaluations in the part (**b**), it is deducible that the radius of lemniscate convexity $r_{\mathcal{L}}^{c}(h_{\rho,\beta})$ is the unique root of equation stated in the part (**c**) of the theorem.

1.2. Janowski starlikeness and Janowski convexity of normalized Wright functions

In the present section we focus on finding out the radii of Janowski starlikeness and Janowski convexity of the normalized Wright functions $f_{\rho,\beta}(z)$, $g_{\rho,\beta}(z)$ and $h_{\rho,\beta}(z)$. It is important to mention here that it is said to be Janowski starlike of a normalized analytic function f if the quantity $\frac{zf'(z)}{f(z)}$ lies in the disc whose diametric end points are $\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$ for $-1\leq B<A\leq1$.

Theorem 3. Let $\rho>0$ and $\beta>0$. Then the following assertions are valid.

a. The radius of Janowski starlikeness $r_{A,B}^{\star}(f_{\rho,\beta})$ is the smallest positive root of the equation

$$\frac{r\lambda'_{\rho,\beta}(r)}{\lambda_{\rho,\beta}(r)}+\beta\left(\frac{A-B}{1+|B|}\right)=0$$

b. The radius of Janowski starlikeness $r_{A,B}^{\star}(g_{\rho,\beta})$ is the smallest positive root of the equation

$$\frac{r\lambda'_{\rho,\beta}(r)}{\lambda_{\rho,\beta}(r)}+\frac{A-B}{1+|B|}=0.$$

c. The radius of Janowski starlikeness $r_{A,B}^{\star}(h_{\rho,\beta})$ is the smallest positive root of the equation

$$\frac{\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r})}{\lambda_{\rho,\beta}(\sqrt{r})}+2\frac{A-B}{1+|B|}=0.$$

Proof. In order to ascertain the radius of Janowski starlikeness of the normalization $f_{\rho,\beta}(z)$ of $\phi(\rho,\beta,\cdot)$ presented in Eq. (3), we need to find a positive real number $r^\star$ such that

$$\left|\frac{\frac{z f'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)}-1}{A-B\frac{z f'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)}}\right|<1, \qquad \text{for } |z|<r^\star.$$

In light of Eq. (7) and by making use of triangle inequality, it can readily be seen that the inequality

$$\left|\frac{\frac{z f'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)}-1}{A-B\frac{z f'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)}}\right|\le\frac{\frac{1}{\beta}\sum_{n\ge1}\frac{2|z|^2}{\lambda^2_{\rho,\beta,n}-|z|^2}}{A-B-|B|\frac{1}{\beta}\sum_{n\ge1}\frac{2|z|^2}{\lambda^2_{\rho,\beta,n}-|z|^2}}, \text{for } |z|<\lambda_{\rho,\beta,1}$$

is valid for $\beta>0$ with equality at $z=|z|=r$. It is clear that the above inequality gives

$$\left|\frac{\frac{z f'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)}-1}{A-B\frac{z f'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)}}\right|\le\frac{1-\frac{z f'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)}}{A-B+|B|\left(\frac{z f'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)}-1\right)}. \tag{15}$$

Let $r^\star$ denote the smallest positive root of the equation

$$\frac{r f'_{\rho,\beta}(r)}{f_{\rho,\beta}(r)}+\frac{A-B}{1+|B|}-1=0, \tag{16}$$

then the inequality given in Eq. (15) implies that the function $f_{\rho,\beta}(z)$ is Janowski starlike for $|z|<r^\star$. In order to complete the proof we must show that the above equation has a unique root in the open interval $(0,\lambda_{\rho,\beta,1})$. To achieve our goal, let us take into consideration the function $u_{\rho,\beta}:(0,\lambda_{\rho,\beta,1})\to\mathbb{R}$ defined by

$$u_{\rho,\beta}(r)=\frac{r f'_{\rho,\beta}(r)}{f_{\rho,\beta}(r)}+\frac{A-B}{1+|B|}-1.$$

Since

$$u'_{\rho,\beta}(r)=-\frac{1}{\beta}\sum_{n\ge1}\frac{4r\lambda^2_{\rho,\beta,n}}{\left(\lambda^2_{\rho,\beta,n}-r^2\right)^2}<0$$

the function $u_{\rho,\beta}$ is strictly decreasing on $(0,\lambda_{\rho,\beta,1})$. Also we can readily obtain that

$$\lim_{r\nearrow\lambda_{\rho,\beta,1}}u_{\rho,\beta}(r)=-\infty \text{ and } \lim_{r\searrow0}u_{\rho,\beta}(r)=\frac{A-B}{1+|B|}>0.$$

Therefore, we draw conclusion that the Janowski starlikeness radius of $f_{\rho,\beta}$, denoted by $r^\star_{A,B}(f_{\rho,\beta})$, is the unique zero of Eq. (16) in the open interval $(0,\lambda_{\rho,\beta,1})$ which is desired result.

It is clear that the similar results can also be obtained for the normalizations $g_{\rho,\beta}$ and $h_{\rho,\beta}$ given in Eqs. (4) and (5), respectively for $\beta>0$. Hence, the radii of Janowski starlikeness $r^\star_{A,B}(g_{\rho,\beta})$ and $r^\star_{A,B}(h_{\rho,\beta})$ are the unique roots of the equations stated in the parts (**b**) and (**c**) of the theorem, respectively.

Theorem 4. Let $\rho>0$ and $\beta>0$. Then the following assertions are valid.

a. The radius of Janowski starlikeness $r^c_{A,B}(f_{\rho,\beta})$ is the smallest positive root of the equation

$$\frac{r\Psi''_{\rho,\beta}(r)}{\Psi'_{\rho,\beta}(r)}+\left(\frac{1}{\beta}-1\right)\frac{r\Psi'_{\rho,\beta}(r)}{\Psi_{\rho,\beta}(r)}+\frac{A-B}{1+|B|}=0.$$

b. The radius of Janowski starlikeness $r^c_{A,B}(g_{\rho,\beta})$ is the smallest positive root of the equation

$$\frac{r^2\lambda''_{\rho,\beta}(r)+2r\lambda'_{\rho,\beta}(r)}{\lambda_{\rho,\beta}(r)+r\lambda'_{\rho,\beta}(r)}+\frac{A-B}{1+|B|}=0.$$

c. The radius of Janowski starlikeness $r^c_{A,B}(h_{\rho,\beta})$ is the smallest positive root of the equation

$$\frac{3\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r})-r\lambda''_{\rho,\beta}(\sqrt{r})}{4\lambda_{\rho,\beta}(\sqrt{r})+2\sqrt{r}\lambda'_{\rho,\beta}(\sqrt{r})}+\frac{A-B}{1+|B|}=0.$$

Proof When taking into consideration definition of Janowski convex function in the open disk $\mathbb{D}_r$, for the function $f_{\rho,\beta}$, we deduce that the inequality

$$\left|\frac{\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}}{A-B\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right)}\right|<1$$

must be valid for $|z|<r$. By using Eq. (10), It can be readily seen that for $0\leq\beta<1$ the function $f_{\rho,\beta}(z)$ satisfies the inequality

$$\left|\frac{\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}}{A-B\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right)}\right|\leq\frac{\sum_{n\geq1}\frac{2z^2}{\zeta'^2_{\rho,\beta,n}-z^2}+\left(\frac{1}{\beta}-1\right)\sum_{n\geq1}\frac{2z^2}{\zeta^2_{\rho,\beta,n}-z^2}}{A-B+|B|\left(\sum_{n\geq1}\frac{2z^2}{\zeta'^2_{\rho,\beta,n}-z^2}+\left(\frac{1}{\beta}-1\right)\sum_{n\geq1}\frac{2z^2}{\zeta^2_{\rho,\beta,n}-z^2}\right)},$$

for $|z|<\zeta'_{\rho,\beta,1}$ with $z=|z|=r$. With the help of inequality (12), the above mentioned inequality holds good for $\beta\geq1$ as well. Due to Eq. (10), by putting $|z|$ in place of z, we obtain

$$\left|\frac{\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}}{A-B\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right)}\right|\leq\frac{-\frac{|z|f''_{\rho,\beta}(|z|)}{f'_{\rho,\beta}(|z|)}}{A-B+|B|\frac{|z|f''_{\rho,\beta}(|z|)}{f'_{\rho,\beta}(|z|)}}$$

for $\beta>0$. Therefore, we draw conclusion that the radius of Janowski convexity $r^c_{A,B}(f_{\rho,\beta})$ is the smallest positive root of the equation

$$\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}+\frac{A-B}{1+|B|}=0. \tag{17}$$

In order to complete our proof, we must show that Eq. (17) has a unique root in the open interval $(0,\zeta'_{\rho,\beta,1})$. To achieve our purpose, let us take into consideration the function $u_{\rho,\beta}:\left(0,\zeta'_{\rho,\beta,1}\right)\to\mathbb{R}$, defined by

$$u_{\rho,\beta}(r)=\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}+\frac{A-B}{1+|B|}$$

which is continous on $(0,\zeta'_{\rho,\beta,1})$ and is strictly decreasing in $(0,\zeta'_{\rho,\beta,1})$ since

$$u'_{\rho,\beta}(r)<4r\sum_{n\geq1}\left(\frac{\zeta^2_{\rho,\beta,n}}{\left(\zeta^2_{\rho,\beta,n}-r^2\right)^2}-\frac{\zeta'^2_{\rho,\beta,n}}{\left(\zeta'^2_{\rho,\beta,n}-r^2\right)^2}\right)<0.$$

Here we used tacitly the interlacing property $\zeta^2_{\rho,\beta,n}\left(\zeta'^2_{\rho,\beta,n}-r^2\right)^2<\zeta'^2_{\rho,\beta,n}\left(\zeta^2_{\rho,\beta,n}-r^2\right)^2$ for $r<\sqrt{\zeta_{\rho,\beta,n}\zeta'_{\rho,\beta,n}}$ and $\beta>0$. We can also observe that

$$\lim_{r\searrow0}u_{\rho,\beta}(r)=\frac{A-B}{1+|B|}>0 \text{ and } \lim_{r\nearrow\zeta'_{\rho,\beta,1}}u_{\rho,\beta}(r)=-\infty.$$

Consequently, the radius of Janowski starlikeness of the function $f_{\rho,\beta}$, denoted by $r^c_{A,B}\left(f_{\rho,\beta}\right)$, is the unique positive root of Eq. (17) in the open interval $(0,\zeta'_{\rho,\beta,1})$. This completes the proof.

Since the results given in part (**b**) and part (**c**) can readily be obtained by imitating the same evaluations in the part (**a**) of the theorem and by bearing in mind the infinite product representations of the functions $g'_{\rho,\beta}(z)$ and $h'_{\rho,\beta}(z)$ (see [9]) we omit the proof of part (**b**) and part (**c**).

Conflicts of interest

The authors state that did not have conflict of interests.

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